A DC Programming Framework for Portfolio Selection by Minimizing the Transaction Costs

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Abstract. We consider a single-period portfolio selection problem which consists of minimizing the total transaction cost subject to different types of constraints on feasible portfolios. The transaction cost function is separable, i.e., it is the sum of the transaction cost associated with each trade, but discontinuous. This optimization problem is nonconvex and very hard to solve. We investigate in this work a DC (Difference of Convex functions) programming framework for the solution methods. First, the objective function is approximated by a DC function. Then a DC formulation for the resulting problem is proposed for which two approaches are developed: DCA (DC Algorithm) and a hybridization of Branch and Bound and DCA.

Keywords: portfolio selection, separable transaction cost, DC programming, DCA, Branch and Bound.

1 Introduction

The mean-variance's model proposed by Markowitz [9] in 1952 is known as a basic for the development of various portfolio selection techniques. While the Markowitz' model is a convex program, extended models considering some factors like transaction costs, cardinality constraints, shortselling, buy-in threshold constraints, etc,... are, in most of cases, nonconvex and very difficult to solve. The portfolio optimization problems including transaction costs have been studied by many researchers [1–4].

In [8], the authors studied two alternative models for the problem of single-period portfolio optimization. The first consists of maximizing the expected return, taking transaction costs into account, and subject to different type of constraints on the feasible portfolios. They proposed a heuristic method for solving this model where the transaction cost is separable and discontinuous. The second model deals with minimizing the total nonconvex transaction cost subject to feasible portfolio constraints. The authors claimed that their heuristic method for solving the former model can be adapted to solve the later.

N.T. Nguyen, T. van Do, and H.A. Thi (Eds.): *ICCSAMA 2013*, SCI 479, pp. 31–40.
DOI: 10.1007/978-3-319-00293-4_3

The starting point of our work is the second model introduced in [8]. We consider a little modified model where the constraints include shortselling constraints, limit on expected return, limit on variance, and diversification constraints. The considered transaction cost is assumed to be separable, say the sum of the transaction cost associated with each trade. It is a discontinuous function that results to a difficult nonconvex program.

We investigate DC programming and DCA for designing solution methods to this problem. DC programming and DCA were first introduced by Pham Dinh Tao in 1985 and have been extensively developed since 1994 by Le Thi Hoai An and Pham Dinh Tao in their common works. DCA has been successfully applied to many large-scale nonconvex programs in various domains of applied sciences, to become now classic and popular (see e.g. [5, 7, 6] and references therein). We first approximate the discontinuous nonconvex objective function by a DC function and then develop DCA for tackling the resulting DC problem. For globally solving the original problem, we propose a hybrid algorithm that combines DCA and a Branch-and-Bound (B&B) scheme. DCA is used for solving the DC approximation problem to compute good upper bounds in the B&B algorithm. Lower bounds are obtained by solving relaxation problems which consist of minimizing a linear function under linear and convex quadratic constraints.

The rest of this paper is organized as follows. In the next section, we describe the considered portfolio problem and its mathematical formulation. Section 3 is concerned with the DC approximation of the considered problem and the description of DCA for solving it. The hybrid Branch and Bound - DCA algorithm is presented in Section 4 while some conclusions are included in the last section.

2 Problem Description and Mathematical Formulation

Consider an investment portfolio that consists of holdings in some or all of n assets.

The current holdings in each asset are $w = (w_i, \dots, w_n)^T$. The total current wealth is then $\mathbf{1}^T w$, where $\mathbf{1}$ is a vector with all entries equal to one. The amount transacted in asset i is x_i, with $x_i > 0$ for buying, $x_i < 0$ for selling and $x = (x_1, \dots, x_n)^T$ is a portfolio selection. After transactions, the adjusted portfolio is $w + x$.

The adjusted portfolio $w+x$ is held for a fixed period time. At the end of that period, the return on asset i is the random variable a_i. We assume knowledge of the first and the second moments of the joint distribution of $a = (a_1, \dots, a_n)$,

$$\mathbf{E}(a) = \bar{a}, \qquad \mathbf{E}(a - \bar{a})(a - \bar{a})^T = \Sigma.$$

A riskless asset can be included, in which case the corresponding $\bar{a}_i$ equal to its return and the i-th row and column of Σ are zero.

The wealth at the end of the period is a random variable, $W = a^T(w + x)$ with expected value and variance given by

$$\mathbf{E}W = \bar{a}^T(w + x), \qquad \mathbf{E}(W - \mathbf{E}W)^2 = (w + x)^T \Sigma (w + x). \tag{1}$$

We consider the problem of minimizing the total transaction costs subject to portfolio constraints:

$$\begin{cases} \min \ \phi(x) \\ \text{s.t. } \bar{a}(w+x) \geq r_{\min}, \\ \qquad w+x \in \mathcal{S}, \end{cases} \tag{2}$$

where $r_{\min}$ is the desired lower bound on the expected return and $\mathcal{S} \subseteq \mathbb{R}^n$ is the portfolio constraint set.

The portfolio constraint set $\mathcal{S}$ can be defined from the following convex constraints:

1. *Shortselling constraints:* Individual bounds s_i on the maximum amount of shortselling allowed on asset i are

$$w_i + x_i \geq -s_i, \quad i = 1, \ldots, n. \tag{3}$$

   If shortselling is not permitted, the s_i are set to zero. Otherwise, $s_i > 0$.
2. *Variance:* The standard deviation of the end period wealth W is constrained to be less than $\sigma_{\max}$ by the convex quadratic inequality

$$(w+x)^T \Sigma (w+x) \leq \sigma_{\max}^2. \tag{4}$$

   ((4) is a *second-order cone constraint*).
3. *Diversification constraints:* Constraints on portfolio diversification can be expressed in terms of linear inequalities and therefore are readily handled by convex optimization. Individual diversification constraints limit the amount invested in each asset i to a maximum of p_i,

$$w_i + x_i \leq p_i, \quad i = 1, \ldots, n. \tag{5}$$

   Alternatively, we can limit the fraction of the total wealth held in each asset,

$$w_i + x_i \leq \lambda_i \mathbf{1}^T (w+x), \quad i = 1, \ldots, n. \tag{6}$$

   They are convex inequality constraints on x.

Transaction costs can be used to model a number of costs, such as brokerage fee, bid-ask spread, taxes or even fund loads. In this paper, the transaction costs $\phi(x)$ is defined by

$$\phi(x) = \sum_{i=1}^{n} \phi_i(x_i), \tag{7}$$

where ϕ_i is the transaction cost function for asset i. We will consider a simple model that includes fixed plus linear costs. Let β_i be the fixed costs common associated with buying and selling asset i. The fixed plus linear transaction cost function is given by

$$\phi_i(x_i) = \begin{cases} 0 & \text{if } x_i = 0, \\ \beta_i - \alpha_i^1 x_i & \text{if } x_i < 0, \\ \beta_i + \alpha_i^2 x_i & \text{if } x_i > 0. \end{cases} \tag{8}$$

The function ϕ is nonconvex, unless the fixed costs are zero.

We develop below two approaches based on DC programming and DCA for solving the problem (2) with $\mathcal{S}$ being defined in (3) - (6) and ϕ being given in (7), (8).

3 DC Programming and DCA for Solving (2)

3.1 DC Approximation Problem

Let C be the feasible set of (2). Since ϕ is discontinuous, we will construct a DC approximation of ϕ. We first compute upper bounds u_i^0 and lower bounds l_i^0 for variables x_i by solving $2n$ convex problems:

$$\min\{x_i \ : \ x \in C\} \quad (LB_i), \qquad \max\{x_i \ : \ x \in C\} \quad (UB_i). \tag{9}$$

Let $R_0 = \prod_{i=1}^{n} [l_i^0, u_i^0]$. The problem (2) can be rewritten as

$$\omega = \min \left\{ \phi(x) = \sum_{i=1}^{n} \phi_i(x_i) \ : \ x \in C \cap R_0 \right\}. \tag{P}$$

For each $i = 1, \ldots, n$, let $\epsilon_i > 0$ be a sufficiently small number chosen as follows:

$$\begin{cases} \epsilon_i < \min\{-l_i^0, u_i^0\} & \text{if } l_i^0 < 0 < u_i^0, \\ \epsilon_i < u_i^0 & \text{if } l_i^0 = 0 < u_i^0, \\ \epsilon_i < -l_i^0 & \text{if } l_i^0 < u_i^0 = 0. \end{cases}$$

Consider the functions $\overline{\phi_i}, \psi_i : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$\overline{\phi_i}(x_i) = \begin{cases} \beta_i - \alpha_i^1 x_i, & x_i \leq 0 \\ \beta_i + \alpha_i^2 x_i, & x_i \geq 0 \end{cases}, \qquad \psi_i(x_i) = \begin{cases} -c_i^1 x_i, & x_i \leq 0 \\ c_i^2 x_i, & x_i \geq 0 \end{cases},$$

where $c_i^j = \left(\frac{\beta_i}{\epsilon_i} + \alpha_i^j \right)$, $j = 1, 2$. By definition, $\overline{\phi_i}, \psi_i$ are convex functions. Then, a DC approximation function f of ϕ can be

$$f(x) = \sum_{i=1}^{n} f_i(x_i), \tag{10}$$

where $f_i(x_i) = g_i(x_i) - h_i(x_i)$ with g_i, h_i being determined by

- $g_i(x_i) = 0$, $h_i(x_i) = -\beta_i + \alpha_i^1 x_i$ if $l_i^0 < u_i^0 < 0$;
- $g_i(x_i) = 0$, $h_i(x_i) = -\beta_i - \alpha_i^2 x_i$ if $0 < l_i^0 < u_i^0$;
- $g_i(x_i) = 0$, $h_i(x_i) = -\min\{-c_i^1 x_i, \beta_i - \alpha_i^1 x_i\}$ if $l_i^0 < u_i^0 = 0$;
- $g_i(x_i) = 0$, $h_i(x_i) = -\min\{c_i^2 x_i, \beta_i + \alpha_i^2 x_i\}$ if $0 = l_i^0 < u_i^0$;
- and if $l_i < 0 < u_i$:

$$g_i(x_i) = \overline{\phi_i}(x_i) + \psi_i(x_i) = \begin{cases} \beta_i - (\alpha_i^1 + c_i^1)x_i & \text{if } x_i \le 0 \\ \beta_i + (\alpha_i^2 + c_i^2)x_i & \text{if } x_i \ge 0 \end{cases},$$

$$h_i(x_i) = \max\{\overline{\phi_i}(x_i), \psi_i(x_i)\} = \begin{cases} -c_i^1 x_i & \text{if } x_i \le -\epsilon_i \\ \beta_i - \alpha_i^1 x_i & \text{if } -\epsilon_i \le x_i \le 0 \\ \beta_i + \alpha_i^2 x_i & \text{if } 0 \le x_i \le \epsilon_i \\ c_i^2 x_i & \text{if } x_i \ge \epsilon_i. \end{cases}$$

It is easy to show that for all cases, g_i, h_i are convex polyhedral functions over $\mathbb{R}$. Therefore, with $g(x) = \sum_{i=1}^{n} g_i(x_i)$ and $h(x) = \sum_{i=1}^{n} h_i(x_i)$, $g - h$ is a DC decomposition of f. In addition,

- $\min\{f(x) \ : \ x \in C \cap R_0\} \le \min\left\{\phi(x) = \sum_{i=1}^{n} \phi_i(x_i) \ : \ x \in C \cap R_0\right\}$.
- For each i, the smaller value of ϵ_i, the better approximation of f_i to ϕ_i over $[l_i^0, u_i^0]$.

The problem (P) with ϕ being replaced by f,

$$\mu = \min\{f(x) = g(x) - h(x) \ : \ x \in C \cap R_0\} \qquad (P_{dc})$$

is a DC approximation problem of (P). We will investigate a DCA scheme for solving this problem.

3.2 DCA for Solving (P_{dc})

DC Programming and DCA. For a convex function θ, the subdifferential of θ at $x_0 \in \text{dom}\theta := \{x \in \mathbb{R}^n : \theta(x_0) < +\infty\}$, denoted by $\partial\theta(x_0)$, is defined by

$$\partial\theta(x_0) := \{y \in \mathbb{R}^n : \theta(x) \ge \theta(x_0) + \langle x - x_0, y\rangle, \forall x \in \mathbb{R}^n\},$$

and the conjugate θ^* of θ is

$$\theta^*(y) := \sup\{\langle x, y\rangle - \theta(x) : x \in \mathbb{R}^n\}, \quad y \in \mathbb{R}^n.$$

A general DC program is that of the form:

$$\alpha = \inf\{F(x) := G(x) - H(x) \,|\, x \in \mathbb{R}^n\}, \qquad (11)$$

where G, H are lower semi-continuous proper convex functions on $\mathbb{R}^n$. Such a function F is called a DC function, and $G - H$ a DC decomposition of F while G and H are the DC components of F. Note that, the closed convex constraint $x \in C$ can be incorporated in the objective function of (11) by using the indicator function on C denoted by χ_C which is defined by $\chi_C(x) = 0$ if $x \in C$, and $+\infty$ otherwise.

A point x^* is called a *critical point* of $G - H$, or a generalized Karush-Kuhn-Tucker point (KKT) of (P_{dc})) if

$$\partial H(x^*) \cap \partial G(x^*) \neq \emptyset. \tag{12}$$

Based on local optimality conditions and duality in DC programming, the DCA consists in constructing two sequences $\{x^k\}$ and $\{y^k\}$ (candidates to be solutions of (11) and its dual problem respectively). More precisely, each iteration k of DCA approximates the concave part $-H$ in (11) by its affine majorization (that corresponds to taking $y^k \in \partial H(x^k)$) and minimizes the resulting convex program.

Generic DCA Scheme
Initialization: Let $x^0 \in \mathbb{R}^n$ be an initial guess, $0 \leftarrow k$.
Repeat
- Calculate $y^k \in \partial H(x^k)$
- Calculate $x^{k+1} \in \arg\min\{G(x) - \langle x, y^k\rangle : x \in \mathbb{R}^n\}$ $\quad (P_k)$
- $k + 1 \leftarrow k$
Until convergence of $\{x^k\}$.

It is worth noting that DCA works with the convex DC components G and H but not the DC function F itself (see [5, 6, 10, 11]). Moreover, a DC function F *has infinitely many DC decompositions* which have crucial impacts on the performance (speed of convergence, robustness, efficiency, globality of computed solutions,...) of DCA.

Convergence properties of DCA and its theoretical basis can be found in [5, 6, 10]. For instant, it is important to mention that (for simplify we omit here the dual part)

– DCA is a descent method (the sequences $\{G(x^k) - H(x^k)\}$ is decreasing) without linesearch.
– If the optimal value α of problem (11) is finite and the infinite sequence $\{x^k\}$ is bounded then every limit point $\widetilde{x}$ of the sequence $\{x^k\}$ is a critical point of $G - H$.
– DCA has a linear convergence for general DC programs.
– DCA has a finite convergence for polyhedral DC programs.

The next subsection is devoted to the development of DCA applied on (P_{dc}).

DC Algorithm for Solving the Problem (P_{dc}). According to the generic DCA scheme, at each iteration k, we have to compute a subgradient $y^k \in \partial h(x^k)$ and then solve the convex program of the form (P_k)

$$\min\{g(x) - \langle y^k, x\rangle \ : \ x \in C \cap R_0\} \tag{13}$$

which is equivalent to

$$\min_{x,t} \left\{ \sum_{i=1}^{n} t_i - \langle y^k, x\rangle \ : \ g_i(x_i) \leq t_i, \forall i = 1, \ldots, n, \ x \in C \cap R_0 \right\}. \tag{14}$$

A subgradient $y^k \in \partial h(x^k)$ is computed by

– if $l_i^0 < u_i^0 < 0 : y_i^k = \alpha_i^1$;
– if $0 < l_i^0 < u_i^0 : y_i^k = -\alpha_i^2$;
– if $l_i^0 < u_i^0 = 0 : y_i^k = \alpha_i^1$ if $x_i^k < -\epsilon i, c_i^1$ if $x_i^k > -\epsilon_i, \in [\alpha_i^1, c_i^1]$ if $x_i^k = -\epsilon_i$;
– if $0 = l_i^0 < u_i^0 : y_i^k = -c_i^2$ if $x_i^k < \epsilon_i, -\alpha_i^2$ if $x_i^k > \epsilon_i, \in [-c_i^2, -\alpha_i^2]$ if $x_i^k = \epsilon_i$;
– if $l_i^0 < 0 < u_i^0$:

$$y_i^k = \begin{cases} -c_i^1, & \text{if } x_i^k < -\epsilon_i, \\ \in [-c_i^1, -\alpha_i^1], & \text{if } x_i^k = -\epsilon_i, \\ -\alpha_i^1, & \text{if } -\epsilon_i < x_i^k < 0, \\ \in [-\alpha_i^1, \alpha_i^2], & \text{if } x_i^k = 0, \\ \alpha_i^2, & \text{if } 0 < x_i^k < \epsilon_i, \\ \in [\alpha_i^2, c_i^2], & \text{if } x_i^k = \epsilon_i, \\ c_i^2, & \text{if } x_i^k > \epsilon_i. \end{cases}$$

Hence, DCA applied on (P_{dc}) can be described as follows.

Algorithm 1 (DCA applied on (P_{dc})):

- **Initialization:** Let $x^0 \in \mathbb{R}^n$ and ε be a sufficiently small positive number; iteration $k \longleftarrow 0$.
- **Repeat:**
  - ⋄ Compute $y^k \in \partial h(x^k)$ as indicated above.
  - ⋄ Solving the convex program (14) to obtain x^{k+1}.
  - ⋄ $k \longleftarrow k + 1$
- **Until:** $|f(x^{k+1}) - f(x^k)| \leq \varepsilon$.

4 A Hybrid Branch and Bound-DCA Algorithm

In this section we propose a combined B&B-DCA algorithm to globally solve the problem (P).

As DCA is a descent and efficient method for nonconvex programming, DCA will be used to improving upper bounds for ω in B&B scheme while lower bounds will be provided by solving relaxation problems constructed over the rectangle $R = \prod_{i=1}^{n} [l_i, u_i]$, subsets of R_0, at each iteration.

4.1 Lower Bounding

A lower bound for ϕ on $C \cap R := \prod_{i=1}^{n} [l_i, u_i] \subset R_0$ can be determined by the following way. Let $B_i = [l_i, u_i],\ i = 1, \ldots, n$. A convex underestimator of the objective function ϕ over the domain $C \cap R$ can be chosen as follows (since ϕ is separable):

$$\widetilde{\phi}_R(x) = \sum_{i=1}^{n} \widetilde{\phi}_{B_i}(x_i) \tag{15}$$

where $\widetilde{\phi}_{B_i}(x_i)$ is defined by the following way:

– if $l_i < u_i < 0$, let $\widetilde{\phi}_{B_i}(x_i) = \beta_i - \alpha_i^1 x_i$;
– if $0 < l_i < u_i$, let $\widetilde{\phi}_{B_i}(x_i) = \beta_i + \alpha_i^2 x_i$;
– if $l_i < u_i = 0$, $\widetilde{\phi}_{B_i}(x_i) = \left(\frac{\beta_i}{l_i} - \alpha_i^1\right) x_i$;
– if $0 = l_i < u_i$, let $\widetilde{\phi}_{B_i}(x_i) = \left(\frac{\beta_i}{u_i} + \alpha_i^2\right) x_i$;
– if $l_i < 0 < u_i$,

$$\widetilde{\phi}_{B_i}(x_i) = \begin{cases} \left(\frac{\beta_i}{l_i} - \alpha_i^1\right) x_i, & x_i \leq 0 \\ \left(\frac{\beta_i}{u_i} + \alpha_i^2\right) x_i, & x_i \geq 0. \end{cases}$$

Hence, solving the convex program

$$\eta(R) = \min\{\widetilde{\phi}_R(x) \ : \ x \in C \cap R\} \tag{16}$$

provides a point $x^R \in C$ satisfying

$$\eta(R) = \widetilde{\phi}_R(x^R) \leq \min\{\phi(x) \ : \ x \in C \cap R\},$$

i.e. $\eta(R)$ is a lower bound for ϕ over $C \cap R$.

4.2 Upper Bounding

Since x^R is a feasible solution to (P), $\phi(x^R)$ is an upper bound for the global optimal value ω of (P). To use DCA for finding a better upper bound for ω, we will construct a DC approximation problem $\min\{f(x) \ : \ x \in C \cap R\}$ of (P) over $C \cap R$ by the same way mentioned in section 3.1 and launch DCA from x^R for solving the corresponding DC approximation problem. Note that we does not restart DCA at every iteration of B&B scheme but only when $\phi(x^R)$ is smaller than the current upper bound.

4.3 Subdivision Process

Let R_k be the rectangle to be subdivided at iteration k of the B&B algorithm and x^{R_k} be an optimal solution of the corresponding relaxation problem of (P) over $C \cap R_k$. We adopt the following rule of bisection of R_k: Choose an index i_k^* satisfying

$$i_k^* \in \arg\max_i \{\phi_i(x_i^{R_k}) - \widetilde{\phi}_i(x_i^{R_k})\}$$

and subdivide R_k into two subsets:

$$R_{k_1} = \{v \in R_k \ : \ v_{i_k^*} \leq x_{i_k^*}^{R_k}\}, \quad R_{k_2} = \{v \in R_k \ : \ v_{i_k^*} \geq x_{i_k^*}^{R_k}\}.$$

We are now in a position to describe our hybrid algorithm for solving (P).

4.4 Hybrid Algorithm

Algorithm 2 (BB-DCA):

- **Initialization:** Compute the first bounds l_i^0, u_i^0 for variables x_i and the first rectangle $R_0 = \prod_{i=1}^{n} [l_i^0, u_i^0]$. Construct the convex underestimator function ϕ_{R_0} of ϕ over R_0 then solve the convex program

$$\min\{\widetilde{\phi}_{R_0}(x) \ : \ x \in C \cap R_0\} \qquad (R_0 cp)$$

  to obtain an optimal solution x^{R_0} and the optimal value $\eta(R_0)$.
  Launch DCA from x^{R_0} for solving the corresponding DC approximation problem (P_{dc}). Let $\overline{x}^{R_0}$ be a solution obtained by DCA.
  Set $\mathcal{R}_0 := \{R_0\}$, $\eta_0 := \eta(R_0)$, $\omega_0 := \phi(\overline{x}^{R_0})$.
  Set $x^* := \overline{x}^{R_0}$.
- **Iteration** $k = 0, 1, 2, \ldots$:

  $k.1$ Delete all $R \in \mathcal{R}_k$ with $\eta(R) \geq \omega_k$. Let $\mathcal{P}_k$ be the set of remaining rectangles. If $\mathcal{P}_k = \emptyset$ then STOP: x^* is a global optimal solution.

  $k.2$ Otherwise, select $R_k \in \mathcal{P}_k$ such that

$$\eta_k := \eta(R_k) = \min\{\eta(R) \ : \ R \in \mathcal{P}_k\}$$

  and subdivide R_k into R_{k_1}, R_{k_2} according to the subdivision process.

  $k.3$ For each R_{k_j}, $j = 1, 2$, construct relaxation function $\widetilde{\phi}_{R_{k_j}}$, and solve

$$\min\{\widetilde{\phi}_{R_{k_j}}(x) \ : \ x \in C \cap R_{k_j}\} \qquad (R_{k_j} cp)$$

  to obtain $x^{R_{k_j}}$ and $\eta(R_{k_j})$.
  If $\phi(x^{R_{k_j}}) < \omega_k$, i.e., the current upper bound is improved on rectangle R_{k_j} then construct a DC approximation problem for (P) over $C \cap R_{k_j}$ by replacing ϕ with DC function $f_{R_{k_j}}$ and launch DCA from $x^{R_{k_j}}$ for solving

$$\min\{f_{R_{k_j}}(x) = g_{R_{k_j}}(x) - h_{R_{k_j}}(x) \ : \ x \in C \cap R_{k_j}\}. \qquad (R_{k_j} DC)$$

  Let $\overline{x}^{R_{k_j}}$ be a solution obtained by DCA. Let

$$\gamma_k = \min\{\phi(x^{R_{k_j}}), \phi(\overline{x}^{R_{k_j}})\}.$$

  $k.4$ Update ω_{k+1} and the best feasible solution known so far x^*.

  $k.5$ Set $\mathcal{R}_{k+1} = (\mathcal{P}_k \setminus R_k) \bigcup \{R_{k_1}, R_{k_2}\}$ and go to the next iteration.

5 Conclusion

We have rigorously studied the model and solution methods for solving a hard portfolio selection problem where the total transaction cost function is nonconvex. Attempting to use DC programming and DCA, an efficient approach in nonconvex programming, we construct an appropriate DC approximation of the objective function, and then investigate a DCA scheme for solving the resulting DC program. The DCA based algorithm is quite simple: each iteration we have to minimize a linear function under linear and convex quadratic constraints for which the powerful CPLEX solver can be used. To get a global minimizer of the original problem we combine DCA with a Branch and Bound scheme. We propose an interesting way to compute lower bounds that leads to the same type of convex subproblems in DCA, say linear program with additional convex quadratic constraints. In the next step we will implement the algorithms and study the computational aspects of the proposed approaches.

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