An overview of the spectral monodromy of non-seladjoint operators

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Sommaire

Understanding the structure of the spectrum of some classes of non-selfadjoint (semi-)classical operators in the semi classical limit.
We will also make the link with classical results that illuminate the initial quantum problem.

Keywords : Non-selfadjoint, integrable system, pseudo-differential operators, asymptotic spectral.

Why studies of non-selfadjoint operators ?

Integrable systems

The phase space is a modeled by a symplectic manifold (M, ω) of $2n-$dimensions. For example $M = T^*X$, where X is a manifold. Let $f \in C^\infty(M)$ be an Hamiltonian. We define the Hamiltonian vector field χ_f and the Hamiltonian flow associated to f by

$$\omega(\chi_f, \cdot) = -df(\cdot),$$

$$\frac{dm(t)}{dt} = \chi_f(m), m \mid_{t=0} = m_0,$$

where m_0 is a given point of M.

We say that

$$F = (f_1, \dots, f_n) : M \to \mathbb{R}^n \tag{1.1}$$

is an integrable system if $\{f_i, f_j\} = 0$ with respect to the Poisson bracket on M.

Let U be an open subset with compact closure of the set of all regular values of F.

Theorem (Angle-action theorem)

Let $c \in U$, and Λ_c be a compact regular leaf of the fiber $F^{-1}(c)$. Then there exists an open neighborhood V of Λ_c in M such that $F\mid_V$ defines a smooth locally trivial fibre bundle onto an open neighborhood $B^c \subset U$ of c, whose fibres are invariant Lagrangian $n-$tori, called Liouville tori. Moreover, there exists a symplectic diffeomorphism κ,

$$\kappa = (x, \xi) : V \to \mathbb{T}^n \times A,$$

with $A \subset \mathbb{R}^n$ is an open subset, such that $F \circ \kappa^{-1}(x, \xi) = \varphi(\xi)$ for all $x \in \mathbb{T}^n$, and $\xi \in A$, and here $\varphi : A \to \varphi(A) = B^c$ is a local diffeomorphism. We call (x, ξ) local angle-action variables near Λ_c and (V, κ) an local angle-action chart.

Classical Monodromy by J. Duistermaat(1980)

The (linear) classical monodromy is as a bundle $H_1(\Lambda_c, \mathbb{Z}) \to c \in U$, associated with a cocycle, denoted by $[\mathcal{M}_{cl}]$ in $\check{H}^1(U, GL(2, \mathbb{Z}))$ of transition functions :

$$\{{}^t\big(d((\varphi^i)^{-1} \circ \varphi^j)\big)^{-1}\}.$$

Quantification

A symbol $a(\cdot; h)$ is associated with a linear operator (in general unbounded) A_h on $L^2(\mathbb{R}^n)$, obtained from the $h-$Weyl-quantization by the integral :

$$(A_h u)(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(x-y)\xi} a(\frac{x+y}{2}, \xi; h) u(y) dy d\xi. \tag{1.2}$$

Through this talk, we always assume that the symbols admit a classical asymptotic expansion in integer powers of h.

The leading term in this expansion is called the principal symbol of operator.

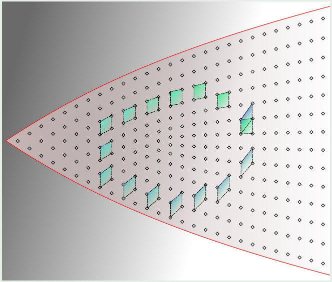
Some examples

1. $M = \mathbb{R}^2_{(x,\xi)}$.
   a) $\xi \mapsto \frac{h}{i}\frac{\partial}{\partial x}$.
   b) $x\xi \mapsto \frac{h}{i}(\frac{1}{2} + x\frac{\partial}{\partial x})$.
2. $M = T^*\mathbb{R}^2_{(x,\xi)}$.
   a) $\xi_1 + \xi_2 \mapsto \frac{h}{i}(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2})$.
   b) $\xi_1\xi_2 \mapsto -h^2 x\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}$.
   c) $H = \frac{\xi_1^2+\xi_2^2}{2m} + V(x) \mapsto \widehat{H} = \frac{-h^2}{2m}\Delta + V$.

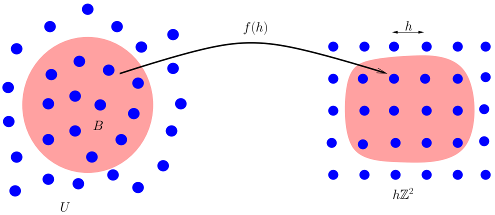
Fundamental examples of integrable systems

Example

1. The harmonic oscillator with trivial monodromy
2. The classic spherical pendulum with non-trivial monodromy



Locally (or even micro-locally) the spectrum of a classical non-selfadjoint operator has the structure of a deformed lattice.



A local chart of spectrum

$\Rightarrow$ Can the spectrum have a globally lattice structure ? How these local lattices are glued ?

Quantum monodromy

An integrable quantum system is given n commuting selfadjoint h-pseudodifferential operators.
Here $n = 2$ for simplicity.

Joint spectrum of an integrable quantum system

The joint spectrum, denoted by $\sigma_J(P_1, P_2)$, is defined by

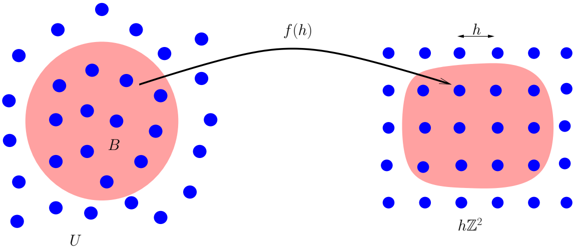
$$\sigma_J(P_1, P_2) = \{(E_1(h), E_2(h)) \in \mathbb{R}^2 | \bigcap_{j=1}^{2} Ker(P_j - E_j(h)) \neq \emptyset\}.$$

Let $\Sigma(h) = \sigma_J(P_1, P_2) \cap U$. Here U is a open bounded subset of regular values of the map $p := (p_0^{(1)}, p_0^{(2)}) : T^*X \to \mathbb{R}^2$.

Theorem (Colin de Verdière and Charbonnel, 98)

For h sufficiently small, $\Sigma(h)$ is discrete, composed of simple joint eigenvalues, and satisfies locally : there exists an invertible symbol of order zero $f(\cdot; h) : B \to \mathbb{R}^2$, from any small ball $B \subset U$ in $\mathbb{R}^2$, sending $\Sigma(h)$ into $h\mathbb{Z}^2$, with modulo $\mathcal{O}(h^\infty)$.

$$\lambda(h) \in \Sigma(h) \cap B + \mathcal{O}(h^\infty) \Leftrightarrow f(\lambda(h); h) \in h\mathbb{Z}^2 + \mathcal{O}(h^\infty).$$



Let $\{B_j\}_{j\in\mathcal{J}}$ be a locally finite covering of U.

Theorem (Vu Ngoc, 99)

On $B_i \cap B_j \neq \emptyset$, the transition maps are in the integer affine group $GA(n, \mathbb{Z})$

$$\left(\frac{f_j(h)}{h}\right) \circ \left(\frac{f_i(h)}{h}\right)^{-1} = A_{ij} + \mathcal{O}(h^\infty),$$

here $A_{ij} \in GA(n, \mathbb{Z})$, independent of h.

Then the quantum monodromy is defined as the 1-cocycle $\{A_{ij}\}$, modulo-coboundary in the Čech cohomology $\check{H}^1(U, GA(n, \mathbb{Z}))$.
It is also the product of transition maps along a closed loop, modulo by conjugation (the holonomy).

Let $P(h)$ be a normal operator and classical of order zero. We can write $P(h) = P_1(h) + iP_2(h)$ with $P_1 = Re(P), P_2 = Im(P)$ (they are selfadjoint, commute).

$$P_1 = \frac{P + P^*}{2}, P_2 = \frac{P - P^*}{2i}, D(P_1) = D(P_2) = D. \qquad (2.3)$$

Let U be some open bounded subset in $\mathbb{C} \cong \mathbb{R}^2$.

Theorem (Phan, 12)

Assume that $\sigma(P(h)) \cap U$ **is discrete**, then we have

$$\sigma(P(h)) \cap U \cong \sigma_J(P_1, P_2) \cap U = \Sigma(h).$$

Therefore, we can define the **affine spectral monodromy** of $P(h)$ as the quantum monodromy of integrable quantum system (P_1, P_2).

We consider a classical operator the form $P_\varepsilon = P(x, hD_x, \varepsilon; h)$, which is a small non-selfadjoint perturbation of a selfadjoint operator with two different assumptions on the classical dynamic of the unperturbed part :

1. completely integrable ;
2. quasi-integrable, together with a globally non-degenerate condition.

We assume that total symbol $P(x, \xi, \varepsilon; h)$ depends smoothly on ε in a neighborhood of $(0, \mathbb{R})$, and that

$$P_{\varepsilon=0} := P \quad \text{is formally selfadjoint.} \tag{3.4}$$

Let p_ε be the principal symbol of P_ε. It is of the form

$$p_\varepsilon = p + i\varepsilon q + \mathcal{O}(\varepsilon^2). \tag{3.5}$$

in the first case, and of the form

$$p_\varepsilon(\lambda) = p_\lambda + i\varepsilon q + \mathcal{O}(\varepsilon^2), \text{ with } p_\lambda = p + \lambda p_1 \tag{3.6}$$

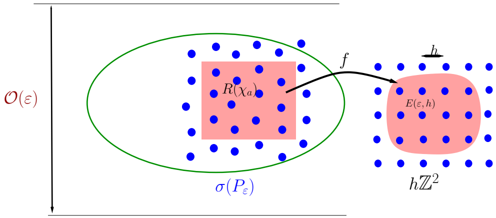
in the second case.
Here p is assumed to be a completely integrable Hamiltonian system.

We assume the **ellipticity condition at infinity**

$$|p_\varepsilon(x,\xi) - E| \geq \frac{1}{C} m(Re(x,\xi)), |\ (x,\xi)\ | \geq C, \qquad (3.7)$$

for some $C > 0$ large enough and m is an order function.
Then the spectrum of P_ε is discrete and contained in a horizontal band of size $\mathcal{O}(\varepsilon)$ of $\mathbb{C}$.

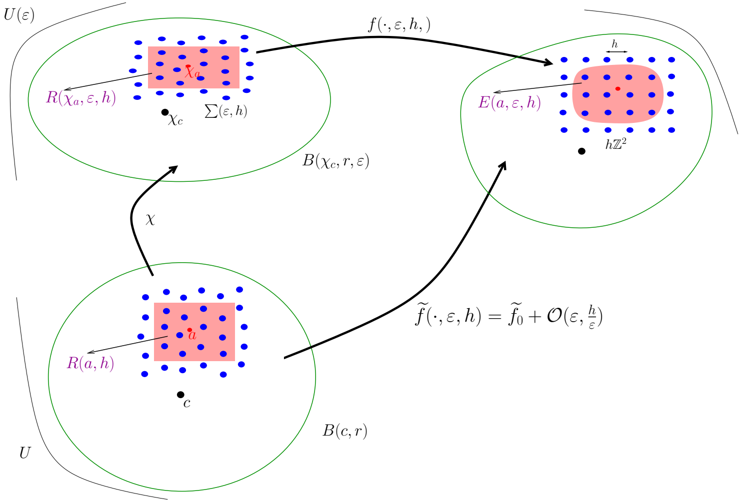
By applying the asymptotic spectral theory [1] of Hitrik, Sjöstrand, and Vu Ngoc, we can obtain asymptotic expansions of eigenvalues of P_ε, located in suitable small complex windows.



A micro-chart of the spectrum of P_ε

Moreover, there is a lot of such windows, which form a family of Cantor type.

Asymptotic pseudo-lattice $(\Sigma(\varepsilon, h), U(\varepsilon))$



An asymptotic pseudo-lattice

We introduce the function

$$\chi : \mathbb{R}^2 \ni u = (u_1, u_2) \mapsto \chi_u \begin{array}{cl} = & (u_1, \varepsilon u_2) \in \mathbb{R}^2 \\ \cong & u_1 + i\varepsilon u_2 \in \mathbb{C}, \end{array} \tag{3.8}$$

in which we identify $\mathbb{C}$ with $\mathbb{R}^2$.

Let $U \subset \mathbb{R}^2$ be a bounded set, and $U(\varepsilon) = \chi(U)$. For h, ε small enough and $h \ll \varepsilon$, let $\Sigma(\varepsilon, h)$ be a discrete set of $U(\varepsilon)$.

With $a = (E, G)$, we define the rectangle

$$R^{(a)}(\varepsilon, h) = (E+i\varepsilon\, G)+\left[-\frac{h^\delta}{\mathcal{O}(1)}, \frac{h^\delta}{\mathcal{O}(1)}\right]+i\varepsilon\left[-\frac{h^\delta}{\mathcal{O}(1)}, +\frac{h^\delta}{\mathcal{O}(1)}\right]. \tag{3.9}$$

Definition

We say that $(\Sigma(\varepsilon, h), U(\varepsilon))$ *is an asymptotic pseudo-lattice if : for any small parameter* $\alpha > 0$, *there exists a set of good values in* $\mathbb{R}^2$, *denoted by* $\mathcal{G}(\alpha)$ *of full measure in the sense :*

$$\mid {}^{C}\mathcal{G}(\alpha) \cap I \mid \leq C\alpha \mid I \mid,$$

with a constant $C > 0$, *for any domain* $I \subset \mathbb{R}^2$.
For every $c \in U$, *there exists a open subset* $B^c \subset U$ *around* c *such that for every good value* $a \in U^c \cap \mathcal{G}(\alpha)$, *there is a adapted good rectangle* $R^{(a)}(\varepsilon, h) \subset B^c(\varepsilon)$ *of the form (3.9), and a smooth local diffeomorphism* $f = f(\cdot; \varepsilon, h)$ *which sends* $R^{(a)}(\varepsilon, h)$ *on its image, satisfying*

$$\Sigma(\varepsilon, h) \cap R^{(a)}(\varepsilon, h) \ni \mu \mapsto f(\mu; \varepsilon, h) \in h\mathbb{Z}^2 + \mathcal{O}(h^\infty). \quad (3.10)$$

Moreover, the function $\widetilde{f} := f \circ \chi$, with χ defined by (3.8), admits an asymptotic expansion in $(\varepsilon, \frac{h}{\varepsilon})$ for the $C^{\infty}-$topology, such that its leading term $\widetilde{f_0}$ is a diffeomorphism, independent of α, locally defined on the whole B^c and independent of the selected good values $a \in B^c$.

We also say that the couple $(f(\cdot;\varepsilon,h), R^{(a)}(\varepsilon,h))$ is a micro-chart, and the family of micro-charts $(f(\cdot;\varepsilon,h), R^{(a)}(\varepsilon,h))$, with all $a \in B^c \cap \mathcal{G}(\alpha)$, is a local pseudo-chart on $B^c(\varepsilon)$ of $(\Sigma(\varepsilon,h), U(\varepsilon))$.

Transition maps

Let $\{B^j\}_{j\in\mathcal{J}}$, here $\mathcal{J}$ is a finite index set, be an arbitrary locally finite covering of U. Then the asymptotic pseudo-lattice $(\Sigma(\varepsilon, h), U(\varepsilon))$ can be covered by the associated local pseudo-charts $\{\big(f_j(\cdot;\varepsilon,h), B^j(\varepsilon)\big)\}_{j\in\mathcal{J}}$. Analyzing transition maps, we have the following result :

Theorem

On each nonempty intersection $B^i \cap B^j \neq \emptyset$, $i, j \in \mathcal{J}$, there exists a unique integer linear map $M_{ij} \in GL(2,\mathbb{Z})$ (independent of h, ε) such that :

$$d\big(\widetilde{f}_{i,0} \circ (\widetilde{f}_{j,0})^{-1}\big) = M_{ij}. \tag{3.11}$$

We can also write (3.11) by

$$d(\widetilde{f}_i) = M_{ij} d(\widetilde{f}_j) + \mathcal{O}(\varepsilon, \frac{h}{\varepsilon}).$$

Monodromy of an asymptotic pseudo-lattice

Definition

The (linear) monodromy of $(\Sigma(\varepsilon, h), U(\varepsilon))$ *is defined as the class of* 1*-cocycle composed of transition maps* $\{M_{ij}\}$*, with modulo coboundary, in the Čech cohomology group* $\check{H}^1(U, GL(2,\mathbb{Z}))$*. We denote it by* $[\mathcal{M}_{sp}]$

It does n't depend on the selected finite covering $\{U^j\}_{j\in\mathcal{J}}$.
The non-triviality of $[\mathcal{M}_{sp}]$ is equivalent to one of its associated holonomy μ,

$$\begin{aligned} \mu : \pi_1(U(\varepsilon)) &\rightarrow GL(2,\mathbb{Z})/\{\sim\} \\ \gamma(\varepsilon) &\mapsto \mu(\gamma(\varepsilon)), \end{aligned} \tag{3.12}$$

where $\{\sim\}$ denote the modulo conjugation.
We call μ the representation of the monodromy $[\mathcal{M}]$.

On the other hand, we can show that

Theorem

The spectrum of P_ε on the domain $U(\varepsilon)$ is an asymptotic pseudo-lattice.

All eigenvalues μ of P_ε in the good rectangle $R^{(E,G)}(\varepsilon, h)$, with modulo $\mathcal{O}(h^\infty)$, are *micro-locally* given by

$$\mu = P\Big(\xi_a + h(k - \frac{\eta}{4}) - \frac{S}{2\pi}; \varepsilon, h\Big) + \mathcal{O}(h^\infty), k \in \mathbb{Z}^2, \quad (3.13)$$

uniformly for small h, ε. Here ξ_a are action coordinates, $S \in \mathbb{R}^2$ is the action integrals, $\eta \in \mathbb{Z}^2$ is the Maslov indices, of the fundamental cycles of Λ_a. $P(\xi; \varepsilon, h)$ is a smooth function admitting an asymptotic expansion in (ξ, ε, h). In particular the $h-$leading term of P is of the form :

$$p_0(\xi, \varepsilon) = p(\xi) + i\varepsilon\langle q\rangle(\xi) + \mathcal{O}(\varepsilon^2). \quad (3.14)$$

So the spectrum in the rectangle is a deformed *micro-lattice*, with horizontal spacing h and vertical spacing εh.

Relationship with the classical monodromy

Theorem

The linear spectral monodromy is the adjoint of the linear classical monodromy

$$[\mathcal{M}_{sp}] = {}^t[\mathcal{M}_{cl}].$$

References

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THANK YOU VERY MUCH !