Maths 17 A

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Hanoi, 11 septembre 2016

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Average Rate of Change/ Instantaneous Rate of Change

The derivative as an Instantaneous Rate of Change

Definition

The derivative of a function f at x, denoted by $f'(x)$, is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided that the limit exists. In this case we say that f is differentiable at x.
Leibniz notation $\frac{df}{dx}$.

Example 1 : Find the difference quotient at $x = 1$ and then $f'(1)$ for the functions $f(x) = x^2$, $g(x) = x^3$.

Example 2 : Find the derivative $f(x) = \frac{1}{x}$ for any $x \neq 0$.

Geometric interpretation of the derivative

Fact

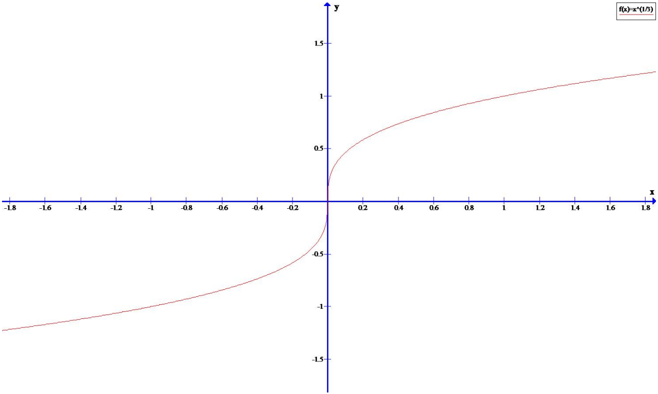
If f is differentiable at $x = c$, then $f'(c)$ is the slope of the tangent line at the point $(c, f(c))$ to the graph of f. The equation of the tangent line is

$$y = f'(c)(x - c) + f(c).$$

Example : Find the tangent line to $f(x) = \frac{1}{x}$ at $x = 2$.

Vertical tangent line

Example 5 page 176 : considering the function $y = x^1/3$ at $x = 0$.
There is not derivative at 0 but $x = 0$ is vertical tangent line to the graph at $(0, 0)$.



Differentiation of fundamental functions

In the following, $u = u(x)$ is a differentiable function of x.

- $(x^\alpha)' = \alpha(x^{\alpha-1})$, and so $(u^\alpha)' = \alpha(u^{\alpha-1})u'$, for any $\alpha \in \mathbb{R}$;
- $(\sin x)' = \cos x$, and so $(\sin u)' = u' \cos u$
- $(\cos x)' = -\sin x$, and so $(\cos u)' = -u' \sin u$
- $(\tan x)' = \frac{1}{\cos^2 x} = \sec^2 x$, and so
  $(\tan u)' = \frac{1}{\cos^2 x} u' = u' \sec^2 u$;
- $(\cot x)' = -\frac{1}{\sin^2 x} = -\csc^2 x$, and so
  $(\cot u)' = -\frac{1}{\sin^2 u} u' = -u' \csc^2 u$;
- $(e^x)' = e^x$, so $(e^u)' = e^u u'$;
  $(a^x)' = (\ln a)a^x$, and so $(a^u)' = (\ln a)a^u u'$, for $0 < a \neq 1$;
- $(\ln x)' = \frac{1}{x}$, and so $(\ln u)' = \frac{u'}{u}$;
  $(\log_a x)' = \frac{1}{(\ln a)x}$, and so $(\ln u)' = \frac{u'}{(\ln a)u}$.

Basic rules of differentiation

$$(u+v)' = u' + v',$$

$$(ku)' = ku', k \in \mathbb{R},$$

$$(uv)' = u'v + uv',$$

$$(\frac{u}{v})' = \frac{u'v - uv'}{v^2}.$$

The chain Rule

Chain rule

If $u = u(x)$ is differentiable at x_0 and $f = f(u)$ is differentiable at $u_0 = u(x_0)$, then the composite function $f \circ u$ is also differentiable at x_0, and

$$(f \circ u)'(x_0) = f'(u_0)u'(x_0).$$

This above expression can be written in Leibniz notation as

$$\frac{d}{dx}[f \circ u(x)] = \frac{df}{du}\frac{du}{dx}.$$

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For any x. We have $f'(u) = (u^3)' = 3u^2$, and $u'(x) = (x^3 - 2x - 1)' = 3x^2 - 2$. Therefore,

$$f'(x) = f'(u).u'(x) = 3u^2(3x^2 - 2) = 3(x^3 - 2x - 1)(3x^2 - 2).$$

Example 2 : find the derivative of radical function

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Example 3 : find the derivative of rational function

$$f(x) = (\frac{x}{x+1})^2, x \neq -1.$$

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Differentiability and Continuity

Theorem

If f is differentiable at c, then it is also continuous at c.

A counter example : $f(x) = |x|$.

Problems

Explaining the following notations

1. Average velocity, instantaneous velocity, speed (page 171) ?
2. Instantaneous population growth rate at time ? The instantaneous per capita population growth rate at time (page 172, 188) ? . Ex 37, 41 page 178. Ex 31, 32, 33 page 242.
3. The rate of Chemical reaction (page 23, 172) ? Ex 29, 30 page 44. Ex 39, 40 page 179.
4. Tilman's model for resource competition (page 173) ? . The specific rate of change of biomass (page 25) ? Example 3 page 188, Ex 35 page 178, Ex 45 page 193.
5. Logitic growth (page 141) ? Ex 62, 63 page 222, Ex 27, 28, 29 page 142.
6. Radioactive (page 220) ? Ex 66-73 page 222.

Considering the equation of the form

$$F(x, y) = 0.$$

With certain suitable conditions, this equation may define (implicitly) y as an function of x, that means $y = f(x)$ such that $F(x, f(x)) = 0$. This function may be even differentiable. Can we express $y'(x)$ (or $\frac{dy}{dx}$) in terms of x and y?

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Example : find $\frac{dy}{dx}$ if $x^4 + y^2 = 1$.

We differentiate both sides of the equation with respect to x, noting that y is function of x we get

$$4x^3 + 2y\frac{dy}{dx} = 0.$$

Now we can solve for $\frac{dy}{dx}$ and find that

$$\frac{dy}{dx} = -2\frac{x^3}{y}.$$

Higher derivatives : n times differentiable

Let f be a function. If there exists f', it is called **the first derivative** and we say that f is **once differentiable**.
The second derivative, denoted by f'' is $f'' = (f')'$ if it exists. Then we say that f is **twice differentiable**.
The same way for **the third derivative**, denoted f''', and for higher derivatives $f^{(4)}$, $f^{(5)}$ etc.
Example : let $f(x) = \sqrt{x}, x \geq 0$. We can rewrite $f(x) = x^{1/2}$, so the first derivative is $f'(x) = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2}$ for $x > 0$.
The higher derivatives

$$f''(x) = (\frac{1}{2})(-\frac{1}{2})x^{-1/2-1} = -\frac{1}{4}x^{-3/2},$$

$$f'''(x) = -\frac{1}{4}(-\frac{3}{2})x^{-3/2-1} = \frac{3}{8}x^{-5/2}, \ldots$$

Linearization (page 235)

Assume that f is differentiable at c, then the tangent line approximation or the linearization of $f(x)$ at $x = c$ is

$$L(x) = f(c) + f'(c)(x - c),$$

and $f(x) \approx L(x)$ for x near c.

4.1.1 Problems, page 177-.
4.2.1 Problems, page 183-.
4.3.3 Problems, page 192-.
4.4.5 Problems, page 208-.
4.5.1 Problems, page 215-.
4.6.1 Problems, page 221-.
4.7.4 Problems, page 233-.

THANK YOU VERY MUCH !