Chapter 11. Systems of differential equations

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Sommaire

Definition

A system of n linear first order differential equations in n unknowns (an $n \times n$ system of linear equations) has the general form :

$$\left\{\begin{array}{rl} x_1' = & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + g_1 \\ x_2' = & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + g_2 \\ \cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n' = & a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + g_n \end{array}\right. \quad (1)$$

where x_i are the functions of the independent variable t to look for ;
the coefficients a_{ij}, and g_i are arbitrary functions of the independent variable t.

If every term g_i is constant zero, then the system is said to be homogeneous.

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Otherwise, it is a nonhomogeneous system if even one of the g_i is nonzero.

The system (1) is most often given in a shorthand format as a matrix-vector equation, in the form :

$$x' = Ax + g,$$

where $A = [a_{ij}]_{n\times n}$, called the coefficient matrix of the system,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ and } g = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}.$$

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Example 1 : the homogeneous system

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Example 2 : the system

$$\begin{cases} x' = & x + y + 2tz + 4t^2 \\ y' = & -2x + y + z + \sin t \\ z' = & 3x - 2e^t y + 5z + \cos t \end{cases}$$

can be written in the form

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2t \\ -2 & 1 & 1 \\ 3 & -2e^t & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 4t^2 \\ \sin t \\ \cos t \end{bmatrix}$$

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Example 3 : a simple epidemic model

$$\begin{cases} S' = & -bSI \\ I' = & bSI - aI \\ R' = & aI \end{cases}$$

We consider the homogenous first order linear system

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{2}$$

or in shorthand $x' = Ax$.

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Fact

If $x^{(1)}$ and $x^{(2)}$ are two solutions to the homogeneous system (2), then

$$C_1x^{(1)} + C_2x^{(2)}, C_1, C_2 \in \mathbb{R}$$

is also a solution to the system.

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is also a solution to the system.

As a consequence, the general solution contains two linearly independent solutions.
As well, the solution will be consisted of some type of exponential functions.

Distinct real eigenvalues

If A has two distinct real eigenvalues λ_1 and λ_2, and suppose that u_1 and u_2 are their respective eigenvectors. Then System (2) has a general solution

$$x = C_1 u_1 e^{\lambda_1 t} + C_2 u_2 e^{\lambda_2 t}, \, C_1, \, C_2 \in \mathbb{R}.$$

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Example 4 : solve the system (2) with

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

(Recall that eigenvalues are the solutions of the characteristic equation of A : $|A - \lambda I| = 0$)

Complex conjugate eigenvalues

If A has two distinct complex conjugate eigenvalues $\lambda_{1,2} = \alpha \pm i\beta$, and suppose that $u + iv$ is an eigenvector corresponding to λ_1.

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Then System (2) has a real-valued general solution

$$x = C_1 e^{\alpha t}[u\cos(\beta t) - v\sin(\beta t)] + C_2 e^{\alpha t}[u\sin(\beta t) + v\cos(\beta t)],$$

for any $C_1, C_2 \in \mathbb{R}$.

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Example 5 : solve the system (2) with

$$A = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}$$

Example 6 : solve the system (2) with

$$A = \begin{bmatrix} -1 & -6 \\ 3 & 5 \end{bmatrix}$$

such that $x(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

Repeated real eigenvalue

Suppose A has a repeated real eigenvalues λ, there are two sub-cases :

(1) If λ has two linearly independent eigenvectors u_1 and u_2, then System (2) has a general solution

$$x = (C_1 u_1 + C_2 u_2)e^{\lambda t}, C_1, C_2 \in \mathbb{R}.$$

(2) If λ only has one linearly independent eigenvector u, then System (2) has a general solution

$$x = [C_1 u + C_2(tu + v)]e^{\lambda t}, C_1, C_2 \in \mathbb{R},$$

where the second vector v is any solution of the nonhomogeneous linear system

$$(A - \lambda I)v = u.$$

Example 7 : solve the system (2) with

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Example 8 : solve the system (2) with

$$A = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix},$$

s.th. $x(0) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Definition

x is called an equilibrium solution to System (1) if it satisfies

$$Ax + g = 0.$$

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