Chapter 8. Differential equations

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Modeling biological situations frequently lead to differential equations.

Definition

A differential equation (**D.E**) is a mathematical equation that relates some function with its derivatives. It contains one (unknown) function of one independent variable and its derivatives.

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Differential equations play a prominent role in many disciplines including engineering, physics, economics, and biology.

We restrict ourselves to first order **separable D.Es**, of the form

$$y' = f(x)g(y), \text{ or } \frac{dy}{dx} = f(x)g(y). \tag{1}$$

Two special cases :

$$\frac{dy}{dx} = f(x), \text{ called Pure time D.Es;}$$

$$\frac{dy}{dx} = g(y), \text{ called Autonomous D.Es.}$$

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How to solve such equations (1) ?

Solution

We separate the variables x and y [assuming $g(y) \neq 0$] and then integrating both sides.

$$\frac{dy}{g(y)} = f(x)dx,$$

$$\int \frac{dy}{g(y)} = \int f(x)dx.$$

Some examples

- Ex1 : assume that $N(0) = N_0 > 0$, solve the equation

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- Ex2 : suppose that the volume $V(t)$ of a cell at time t changes according to the equation
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- Ex3 : solve the D.E.
$$\frac{dy}{dx} = (y+1)x, \text{ where } y(0) = 2.$$
- Ex4 : solve the D.E.
$$\frac{dy}{dx} = 2\frac{y}{x}.$$

- Exponential population growth (page 479).
- Restricted growth : von Bertalanffy equation (page 481).
- The logistic equation (page 486).
- Allometric growth (page 488, optional).
- Homeostasis (page 490, optional).

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Consider the logistic equation

$$\frac{dN}{dt} = r\ N(1 - \frac{N}{K}),\ \text{with } N(0) = N_0 \tag{2}$$

where r and K are positive constants.
The solution is

$$N(t) = \frac{K}{1 + (\frac{K}{N_0} - 1)e^{-rt}}. \tag{3}$$

If $N_0 > 0$, the $N(t) \to K$, the carrying capacity, as $t \to \infty$.
And if $N_0 = 0$, then $N(t) = 0$ for all $t > 0$. Also if if $N_0 = K$, then $N(t) = K$ for all $t > 0$.
We can see from Equation (2) that if $N = K$ or $N = 0$, then $\frac{dN}{dt} = 0$, implying that $N(t)$ is constant. They are called point equilibria.

Definition

A constant $\widehat{y}$ is called an equilibrium of autonomous diff. eqs.

$$\frac{dy}{dx} = g(y) \tag{4}$$

if $g(\widehat{y}) = 0$.

Example : Find the equilibria of the dif. eq. and discuss the behavior of the solutions near these points, using the graph approach.

$$\frac{dy}{dx} = y(2016 - y).$$

Definition

Assume that $\widehat{y}$ is an equilibrium of Equation (4). Then

1. $\widehat{y}$ is locally stable if $g'(\widehat{y}) < 0$, and
2. $\widehat{y}$ is locally unstable if $g'(\widehat{y}) > 0$.

Example : Find the equilibria of the dif. eq. and discuss their stability

$$\frac{dN}{dt} = 2N(1 - \frac{N}{100}).$$

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