Chapter 10. Multivariables calculus

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Content

Let $\mathbb{R}^n$ be the set of all $n-$ tuples $(x_1, x_2, \dots, x_n)$,

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) : x_1 \in \mathbb{R}, x_2 \in \mathbb{R}, \dots, x_n \in \mathbb{R}\}.$$

Definition

Suppose $D \subseteq \mathbb{R}^n$. We write a real-valued function f of n independent variables on D as

$$\begin{aligned} f : D &\longrightarrow \mathbb{R} \\ (x_1, x_2, \dots, x_n) &\mapsto f(x_1, x_2, \dots, x_n) \end{aligned}$$

D is called the domain of f, and the set

$$\{w \in \mathbb{R} : w = f(x_1, x_2, \dots, x_n), (x_1, x_2, \dots, x_n) \in D\}$$

is the range of f.

Example 1 : evaluate the function

$$f(x, y, z) = \frac{x + y}{z^2}$$

at the point $(1, 2, 3)$ and $(-1, 2, 3)$. Give the domain of f.

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$$f(x, y) = \sqrt{4 - x^2 - y^2}.$$

Give and then graph the domain of f in $x - y$ plane. Find the range of f.

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Example 3 The same above question for the function $f = \sqrt{y^2 - x}$.

The graph of a function of two variables

Considering the function $z = f(x, y)$, $(x, y) \in D \subseteq \mathbb{R}$.
Using the Cartesian coordinate system to locate (x, y, z) in 3-dimensional space, we get the graph of the function as a surface in 3-dimensional space.

Example 1 : $z = x + y - 1$.

Example 2 : $z = \sqrt{4 - x^2 - y^2}$. (an upper hemisphere)

Example 3 : $z = 4x^2 + y^2$. (an elliptic paraboloid)

Example 4 : $z = 2x^2 - y^2$. (a hyperbolic paraboloid)

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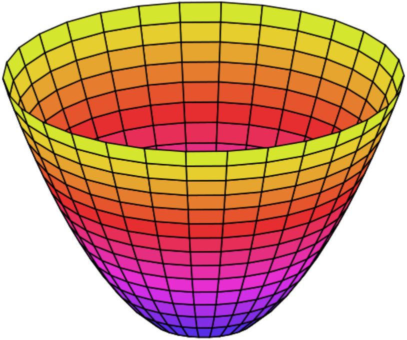
a hemisphere

$$z = \sqrt{4 - x^2 - y^2}$$



an elliptic paraboloid

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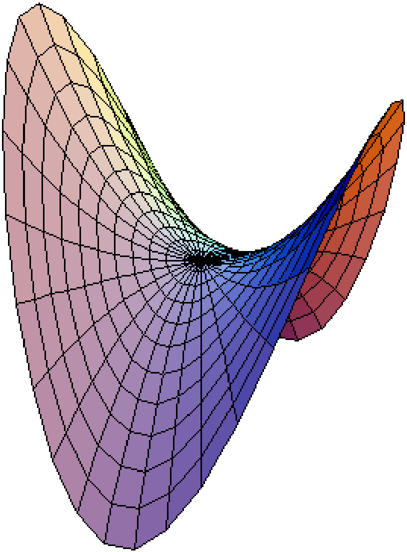
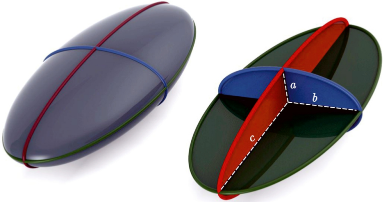


a hyperbolic paraboloid

$$z = 2x^2 - y^2$$



an ellipsoid (like a rugby ball)

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Example : determine level curves of the following functions

$$z = \sqrt{4 - x^2 - y^2}$$

$$z = x^2 + y^2 \text{ (Round incense)}$$

$$z = 2x^2 - y^2$$

Informal definition of limits

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L.$$

Remark (x, y) can approach (x_0, y_0) along any path.

Example 1 :

$$\lim_{(x,y)\to(1,1)} (2x + y) = 2 + 1 = 3.$$

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* Self reading limit laws, page 617.

Remark If (x, y) approaches (x_0, y_0) along two paths, however $f(x, y)$ approaches two different values, then there is not limit.

Example :

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}.$$

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$$f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & \text{for} \quad (x,y) \neq (0,0) \\ 0 & \text{for} \quad (x,y) = (0,0) \end{cases}$$

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Partial derivatives

Definition

Let f is a function of x, y. Then its partial derivatives at (x_0, y_0) are

$$\frac{d}{dx} f(x, y_0)|_{x=x_0} := f_x(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0) \text{ (w.r.t. } x\text{)},$$

$$\frac{d}{dy} f(x_0, y)|_{y=y_0} := f_y(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) \text{ (w.r.t. } y\text{)}$$

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Example : find the partial derivatives of $f(x, y) = x^2 + 2x + y^2 - 1$.

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Example : find the partial derivatives of $f(x, y) = \frac{x+y}{y} + 2x$.

Remark To compute f_x, we treat y as constant, and differentiate f with respect to x.

Higher-order partial derivatives

We can define the second partial derivatives

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}, f_{yy} = \frac{\partial^2 f}{\partial y^2}, f_{xy} = \frac{\partial^2 f}{\partial y \partial x}, f_{yx} = \frac{\partial^2 f}{\partial x \partial y}.$$

Example : $f(x, y) = x^3y^2 + 2x^2y + 1.$

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Example : $f(x, y) = xe^y + 2x.$

The mixed derivative theorem

If the function $f(x, y)$ has the mixed derivatives f_{xy}, f_{yx} which are both continuous in a neighborhood of (x_0, y_0), then we have $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$.

Differentiability

Consider the function $f(x, y)$ and the (x_0, y_0) in the domain of f. Let $\triangle x = x - x_0$, $\triangle y = y - y_0$ and
$\triangle f = f(x, y) - f(x_0, y_0) = f(x_0 + \triangle x, y_0 + \triangle y) - f(x_0, y_0)$.

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The function f is called differentiable at (x_0, y_0) if $\triangle f$ can be expressed in the form

$$\triangle f = \left[\frac{\partial f}{\partial x}(x_0, y_0)\triangle x + \frac{\partial f}{\partial y}(x_0, y_0)\triangle y\right] + \varepsilon_1 \triangle x + \varepsilon_2 \triangle y,$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\triangle x, \triangle y) \to (0, 0)$.

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where ε_1 and $\varepsilon_2 \to 0$ as $(\triangle x, \triangle y) \to (0, 0)$.

In this case, the differential (also called the total differential) of f denoted by df, is defined by

$$df(x_0, y_0) = [\cdots] = f_x dx + f_y dy.$$

Tangent plane

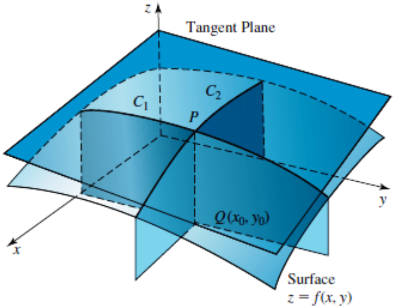

Figure 10.37 The surface $z = f(x, y)$ and its tangent plane at $P = (x_0, y_0, z_0)$.

If there exists the partial derivative $f_x(x_0, y_0)$, $f_y(x_0, y_0)$, then the tangent plane to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0), where $z_0 = f(x_0, y_0)$, has the equation

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Example : find the tangent plane to the surface

$$z = f(x, y) = 4x^2 + y^2$$

at the point $(1, 2, 8)$.

Linearization

We have

$$f(x, y) \approx z_0 + dfx_0, y_0),$$

$$f(x, y) \approx z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$

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Example : find the linearization of $f(x,y) = \ln(x - 2y^2)$ at $(3,1)$ and use it to find an approximation for $f(3.05, 0.95)$.

A vector-valued function is a function from domain $\mathbb{R}^n$ to codomain $\mathbb{R}^m$,

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

$$(x_1, x_2, \ldots, x_n) \mapsto (f_1(x_1, x_2, \ldots, x_n)), \ldots, f_m(x_1, x_2, \ldots, x_n)).$$

Example :

$$f(x, y) = (4x^2 + y^2, 2x + y)$$

$$f(x, y) = (x \sin y, 2x + y, xy)$$

$$f(x, y, z) = (\frac{x}{yz}, x + y + z)$$

Jacobi matrix

The Jacobi matrix or the derivative matrix of f at here $x_0 \in \mathbb{R}^n$ is

$$Df(x_0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} (x_0) \tag{1}$$

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Example : find the Jacobi matrix of the functions

$$f(x, y) = (4x^2 + y^2, 2x + y)$$

$$g(x, y) = (x \sin y, 2x + y, xy).$$