Vietnam National University of Agriculture
Department of Mathematics

April 2, 2017

$\boxed{\text{FINAL EXAM MATH 17B- No.1}}$

Duration 75 minutes

Unauthorized materials

Exercise 1. $\boxed{3.0 \text{ pt}}$ Evaluate the following integrals

a) $\boxed{1.0 \text{ pt}}$ $\int_0^1 x^3\sqrt{1+x^2}dx$.

b) $\boxed{1.0 \text{ pt}}$ $\int x\ln(x+1)dx$.

c) $\boxed{1.0 \text{ pt}}$ $\int_1^\infty \frac{1}{x^{\frac{1}{2}}}dx$ (if it exists).

Exercise 2. $\boxed{2.0 \text{ pt}}$ Draw and find the area of the region bounded by $y = x^2$, $y = \frac{1}{x}$, $y = 4$ in the first quadrant.

Exercice 3. $\boxed{3.0 \text{ pt}}$ Denote by $L(t)$ the length of a fish at time t and assume that the fish grows according the von Bertalanffy equation

$$\frac{dL}{dt} = 2(18 - L), \text{ with } L(0) = 6.$$

a) $\boxed{1.0 \text{ pt}}$ Solve the equation.

b) $\boxed{1.0 \text{ pt}}$ Find the length of the fish at time $t = 1$.

c) $\boxed{1.0 \text{ pt}}$ Find the average length of the fish between $t = 0$ and $t = 1$.

Exercice 4. $\boxed{2.0 \text{ pt}}$ Let $N(t)$ denote the size of a population at time t that satisfies the logistic equation

$$\frac{dN}{dt} = 2N(1 - \frac{N}{1000}) - 100, \text{ for } N \geq 0.$$

Find all equilibria of the equation and determine their stability.

Edited by Quang Sang Phan   Reviewed by

Vietnam National University of Agriculture
Department of Mathematics

April 3, 2017

FINAL EXAM MATH 17B- No.2

Duration 75 minutes

Unauthorized materials

Exercise 1. 3.0 pt Evaluate the following integrals

a) 1.0 pt $\int_1^2 x^3\sqrt{x^2-1}dx$.

b) 1.0 pt $\int x\ln(x-1)dx$.

c) 1.0 pt $\int_1^\infty \frac{1}{x^{\frac{3}{4}}}dx$ (if it exists).

Exercise 2. 2.0 pt Draw and find the area of the region bounded by $y = x^2$, $y = 2 - x$, $y = 0$ in the first quadrant.

Exercice 3. 3.0 pt Denote by $L(t)$ the length of a fish at time t and assume that the fish grows according the von Bertalanffy equation

$$\frac{dL}{dt} = \frac{1}{2}(20 - L), \text{ with } L(0) = 8.$$

a) 1.0 pt Solve the equation.

b) 1.0 pt Find the length of the fish at time $t = 2$.

c) 1.0 pt Find the average length of the fish between $t = 0$ and $t = 2$.

Exercice 4. 2.0 pt Let $N(t)$ denote the size of a population at time t that satisfies the logistic equation

$$\frac{dN}{dt} = 2N(1 - \frac{N}{1200}), \text{ for } N \geq 0.$$

Find all equilibria of the equation and determine their stability.

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Vietnam National University of Agriculture
Department of Mathematics

April 3, 2017

SOLUTION FOR THE FINAL EXAM MATH 17B- No.1
Duration 75 minutes

Unauthorized materials

Exercise 1. $\boxed{3.0 \text{ pt}}$ Evaluate the following integrals

a) $\boxed{1.0 \text{ pt}}$ Let $t = \sqrt{1+x^2}$, then $t^2 = 1 + x^2$, $tdt = xdx$, $x^2 = t^2 - 1$,

$$I_1 = \int_1^{\sqrt{2}} (t^2-1)t^2 dt = \int_1^{\sqrt{2}} (t^4 - t^2)dt = (\frac{t^5}{5} - \frac{t^3}{3}) \mid_1^{\sqrt{2}} = \frac{2\sqrt{2}+2}{15}.$$

b) $\boxed{1.0 \text{ pt}}$ $\int x \ln(x+1)dx$. Let $u = \ln(x+1), v' = x$ then $u' = \frac{1}{x+1}$, $v = \frac{1}{2}x^2$. By IBP we have

$$\begin{aligned} I_2 &= \frac{1}{2}x^2 \ln(x+1) - \int \frac{1}{2}\frac{x^2}{x+1}dx = \frac{1}{2}x^2 \ln(x+1) - \frac{1}{2}\int (x - 1 + \frac{1}{x+1})dx \\ &= \frac{1}{2}x^2 \ln(x+1) - \frac{1}{2}(\frac{1}{2}x^2 - x + \ln|x+1|) + C,\ C \in \mathbb{R} \\ &= \frac{1}{2}x^2 \ln(x+1) - \frac{1}{4}x^2 + \frac{1}{2}x - \frac{1}{2}\ln|x+1| + C,\ C \in \mathbb{R}. \end{aligned}$$

c) $\boxed{1.0 \text{ pt}}$ $\int_1^\infty \frac{1}{x^{\frac{1}{2}}}dx \int_1^\infty x^{-\frac{1}{2}}dx = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = +\infty$, so the integral is not convergent.

Exercise 2. $\boxed{2.0 \text{ pt}}$ The points of intersection of these curves in the first quadrant are $(\frac{1}{4}, 4); (1, 1); (2, 4)$. Then

$$\begin{aligned} A &= \int_{\frac{1}{4}}^1 (4 - \frac{1}{x})dx + \int_1^2 (x^2 - \frac{1}{x})dx \\ &= (4x - \ln|x|)|_{\frac{1}{4}}^1 + (\frac{x^3}{3} - \ln|x|)|_1^2 = \frac{16}{3} - \ln 4 - \ln 2. \end{aligned}$$

Exercise 3. $\boxed{3.0 \text{ pt}}$

a) $\boxed{1.0 \text{ pt}}$ We rewrite the equation in the form $\frac{dL}{L-18} = -2dt$, then

$$\ln|L - 18| = -2t + \ln|C|, C \in \mathbb{R},\ \text{then}\ L = 18 + Ce^{-2t}.$$

The initial condition $L(0) = 6$ leads to $C = -12$. Therefore

$$L(t) = 18 - 12e^{-2t}.$$

b) $\boxed{1.0 \text{ pt}}$ The length of the fish at time $t = 1$ is

$$L(1) = 18 - 12e^{-2} = ...$$

c) $\boxed{1.0 \text{ pt}}$ The average length of the fish between $t = 0$ and $t = 1$ is

$$\frac{1}{1-0}\int_0^1 L(t)dt = \frac{1}{1-0}\int_0^1 (18 - 12e^{-2t})dt = (18t + 6e^{-2t})|_0^1 = 12 + 6e^{-2} = ...$$

Exercice 4. $\boxed{2.0 \text{ pt}}$ The equilibrium points are the solutions of the equation $\frac{dN}{dt} = 0$. Then

$$3N(1 - \frac{N}{1000}) = 0, \text{ so } N = 0 \text{ or } N = 1000.$$

We have $g'(N) = 3(1 - \frac{1}{500}N)$. We check $g'(0) = 3 > 0$ and $g'(1000) = -3 < 0$.
Therefore $N = 0$ is an unstable equilibrium, and $N = 1000$ is a locally stable equilibrium.

VIETNAM NATIONAL UNIVERSITY OF AGRICULTURE
DEPARTMENT OF MATHEMATICS

April 3, 2017

SOLUTION FOR THE FINAL EXAM MATH 17B- No.2
Duration 75 minutes

Unauthorized materials

Exercise 1. $\boxed{3.0 \text{ pt}}$ Evaluate the following integrals

a) $\boxed{1.0 \text{ pt}}$ Let $t = \sqrt{x^2 - 1}$, then $t^2 = x^2 - 1$, $tdt = xdx$, $x^2 = t^2 + 1$,

$$I_1 = \int_0^{\sqrt{3}} (t^2 + 1)t^2 dt = \int_0^{\sqrt{3}} (t^4 + t^2)dt = (\frac{t^5}{5} + \frac{t^3}{3}) \,|_0^{\sqrt{3}} = \frac{42\sqrt{3}}{15}.$$

b) $\boxed{1.0 \text{ pt}}$ $\int x \ln(x + 1)dx$. Let $u = \ln(x - 1), v' = x$ then $u' = \frac{1}{x-1}$, $v = \frac{1}{2}x^2$. By IBP we have

$$\begin{aligned} I_2 &= \frac{1}{2}x^2 \ln(x - 1) - \int \frac{1}{2} \frac{x^2}{x - 1} dx = \frac{1}{2}x^2 \ln(x - 1) - \frac{1}{2} \int (x + 1 + \frac{1}{x - 1})dx \\ &= \frac{1}{2}x^2 \ln(x - 1) - \frac{1}{2}(\frac{1}{2}x^2 + x + \ln|x - 1|) + C, \; C \in \mathbb{R} \\ &= \frac{1}{2}x^2 \ln(x - 1) - \frac{1}{4}x^2 - \frac{1}{2}x - \frac{1}{2} \ln|x - 1| + C, \; C \in \mathbb{R}. \end{aligned}$$

c) $\boxed{1.0 \text{ pt}}$ $\int_1^{\infty} \frac{1}{x^{\frac{3}{4}}} dx \int_1^{\infty} x^{-\frac{3}{4}} dx = \frac{x^{\frac{1}{4}}}{\frac{1}{4}} = +\infty$, so the integral is not convergent.

Exercise 2. $\boxed{2.0 \text{ pt}}$ The points of intersection of these curves in the first quadrant are $(0, 0); (1, 1); (2, 0)$. Then

$$\begin{aligned} A &= \int_0^1 x^2 dx + \int_1^2 (2 - x)dx \\ &= (\frac{x^3}{3})|_0^1 + (2x - \frac{x^2}{2})|_1^2 = \frac{5}{6}. \end{aligned}$$

Exercice 3. $\boxed{3.0 \text{ pt}}$

a) $\boxed{1.0 \text{ pt}}$ We rewrite the equation in the form $\frac{dL}{L-20} = -\frac{1}{2}dt$, then

$$\ln|L - 20| = -\frac{1}{2}t + \ln|C|, C \in \mathbb{R}, \text{ then } L = 20 + Ce^{-\frac{1}{2}t}.$$

The initial condition $L(0) = 8$ leads to $C = -12$. Therefore

$$L(t) = 20 - 12e^{-\frac{1}{2}t}.$$

b) $\boxed{1.0 \text{ pt}}$ The length of the fish at time $t = 2$ is

$$L(2) = 20 - 12e^{-1} = ...$$

c) $\boxed{1.0 \text{ pt}}$ The average length of the fish between $t = 0$ and $t = 2$ is

$$\frac{1}{2-0}\int_0^2 L(t)dt = \frac{1}{2-0}\int_0^2 (20 - 12e^{-\frac{1}{2}t})dt = \frac{1}{2}(20t + 24e^{-\frac{1}{2}t})|_0^2 = 8 + 12e^{-1} = ...$$

Exercice 4. $\boxed{2.0 \text{ pt}}$ The equilibrium points are the solutions of the equation $\frac{dN}{dt} = 0$. Then

$$2N(1 - \frac{N}{1200}) = 0, \text{ so } N = 0 \text{ or } N = 1200.$$

We have $g'(N) = 2(1 - \frac{1}{600}N)$. We check $g'(0) = 3 > 0$ and $g'(1200) = -2 < 0$.
Therefore $N = 0$ is an unstable equilibrium, and $N = 1200$ is a locally stable equilibrium.