Chapter 9. Linear algebra and analytic geometry

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September 11, 2018

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Definition

A matrix is a rectangular array (m rows and n columns) of numbers, arranged in rows and columns

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{1}$$

The elements a_{ij} are called entries. The size of a matrix is $m \times n$ (read "m by n"), and A is an $m \times n$ matrix.
We write also the shorthand notation $A = [a_{ij}]_{m \times n}$, or $A = [a_{ij}]$.

We say two matrices are equal if they have the same size and the same corresponding entries.

Some special matrixes:

- square matrix;

- identity matrix I_n;

- the zero matrix θ;

- column vector, row vector;

- upper/ lower triangular matrix;

- diagonal matrix.

Basic matrix operations

Transposition

The transpose of an $m \times n$ matrix A is the $n \times m$ matrix A' (also denoted A^t) formed by turning rows into columns and vice versa:

Example:

$$\begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & -2 \end{bmatrix}' = \begin{bmatrix} 2 & -1 \\ 1 & 3 \\ 0 & 2 \end{bmatrix},$$

$$\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix}'.$$

Addition of matrix of equal size

Suppose $A = [a_{ij}]$ and $B = [b_{ij}]$ are two $m \times n$ matrixes. Then $C = A + B = [c_{ij}]_{m\times n}$, with $c_{ij} = a_{ij} + b_{ij}$ for $1 \leq i \leq m$, $1 \leq j \leq n$.

Scalar multiplication

Let $A = [a_{ij}]_{m\times n}$. Then we define $kA = [ka_{ij}]_{m\times n}$ for any number (scalar) k.

Example: Let the following matrixes

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & -2 \end{bmatrix},\ B = \begin{bmatrix} -1 & 0 & 2 \\ 2 & 1 & 1 \end{bmatrix},\ c = \begin{bmatrix} 4 & 2 & -2 \\ 1 & 1 & 1 \end{bmatrix}.$$

Find $A + B$, $A - B + 2C$.

Matrix multiplication

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times \ell}$. Then their matrix product $C = AB = [c_{ij}]_{m\times \ell}$, with

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

(c_{ij} is given by the product of the ith row of A and the jth column of B)

Example:

$$\begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -13 & 12 \end{bmatrix}.$$

Note that the matrix multiplication does not commute.

Some properties

1. $A\theta = \theta$, $\theta A = \theta$.
2. $(A + B)C = AC + BC$, $A(B + C) = AB + AC$.
3. $(AB)C = A(BC)$.
4. $(AB)' = B'A'$.

Inverse matrixes

Let A be an $n \times n$ square matrix. If there exists an $n \times n$ square matrix B such that $AB = BA = I_n$, then we say that A is invertible or nonsingular and B is called the inverse matrix of A, denoted by A^{-1}.

Example: Let $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, then we can check that

$$A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

Some properties

1. A^{-1} if exists is unique.
2. $(A^{-1})^{-1} = A$.
3. $(AB)^{-1} = B^{-1}A^{-1}$.

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$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots & \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = b_m \end{cases} \tag{2}$$

Let $A = [a_{ij}]_{m\times n}$, called the coefficient matrix of the system,

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \; x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Matrix representation of linear systems: $Ax = b$.

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Transforming this system into an equivalent system in upper triangular form by using the following basic operations:

1. Multiplying an equation by a nonzero constant;
2. Adding one equation to another;
3. Rearranging the order of the equations.

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Example 1: solve the system

$$\begin{cases} x + y + 2z & = 4 \\ -2x + y + z & = 0 \\ 3x - 2y + 5z & = 6 \end{cases}$$

Example 2: solve the system

$$\left\{\begin{array}{ll} x+y+2z & =4 \\ -2x+y+z & =0 \\ x+4y+7z & =12 \end{array}\right. \tag{3}$$

Inverse matrixes of size 2×2

Example: find the inverse matrix of A (in the example 9)

$A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

Set $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & a_{22} \end{bmatrix}$. We need $AB = BA = I$, therefore

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This leads to the following set of equations:

$$\begin{cases} 2b_{11} + 5b_{21} & = 1 \\ 1b_{11} + 3b_{21} & = 0 \\ 2b_{12} + 5b_{22} & = 0 \\ 1b_{11} + 3b_{22} & = 1 \end{cases}$$

and so we get $b_{11} = 3, b_{21} = -1, b_{12} = -5, b_{22} = 2$.

Hence $B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = A^{-1}$. (check again $AB = BA = I$)

Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

We define the determinant of A, denoted by $det(A)$ (or $|A|$) by

$$det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

Theorem

Suppose A is an $n \times n$ matrix. A is invertible iff $det(A) \neq 0$. In particular with $n = 2$,

$$A^{-1} = \frac{1}{det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Example: let the following matrixes $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$,

$B = \begin{bmatrix} -1 & 3 & 5 \\ 2 & 4 & -2 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 2 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}$, and $b = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Find the matrixes X, Y and Z satisfy $AX = b$, $AY = B$, $ZA = C$.

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Find the matrixes X, Y and Z satisfy $AX = b$, $AY = B$, $ZA = C$.

Check that $det(A) \neq 0$, therefore A is invertible. Computing A^{-1} and then $X = A^{-1}b$, $Y = A^{-1}B$, $Z = CA^{-1}$.

Points and vectors in higher dimensions

Recall that $\mathbb{R}^n$ is the set of points

$$\mathbb{R}^n = \{X = (x_1, x_2, \ldots, x_n) : x_1 \in \mathbb{R}, x_2 \in \mathbb{R}, \ldots, x_n \in \mathbb{R}\}.$$

We use usually a Cartesian coordinate system to represent these points.

A vector is a column vector, $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

Then the vector x has a graphical representation as a directed segment (an arrow) $\overrightarrow{OX}$ from the origin $O = (0, 0, \ldots, 0)$ to the endpoint $X = (x_1, x_2, \ldots, x_n)$. It has a direction and a magnitude (a length).
The length of x is

$$|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

$n = 2$: the polar coordinate system

Let $\alpha \in [0, 2\pi)$ be the angle between the positive x_1-axis and the vector x, measured counterclockwise. It determine the direction of x, $\tan\alpha = \frac{x_2}{x_1}$.
We denote $r = |x|$, then, we have $x_1 = r\cos\alpha$ and $x_2 = r\sin\alpha$. The vector x can be written as

$$x = \begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix}$$

The couple (r, θ) is called the polar coordinates of x. We have an alternative way of representing vectors in plane.
Example: Find the the polar coordinates for the vectors
$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$.

Example: find the representation of the vectors in Cartesian coordinates: $r = 2, \alpha = 30^\circ$; $r = 3, \alpha = 150^\circ$.

The dot product

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, and $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$. We define their dot product by

$$x \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Proposition

$$x \cdot y = |x||y| \cos\theta,$$

where θ is the angle between x and y.
Hence x and y are perpendicular iff $x \cdot y = 0$.

(Some examples)

Equation of hyperplanes

Find an equation of the hyperplane through a fixed point $(x_1^*, x_2^*, \ldots, x_n^*)$ that is perpendicular to the vector $u = [u_1, u_2, \ldots, u_n]'$.

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Find an equation of the hyperplane through a fixed point $(x_1^*, x_2^*, \ldots, x_n^*)$ that is perpendicular to the vector $u = [u_1, u_2, \ldots, u_n]'$.

$$(x_1 - x_1^*)u_1 + (x_2 - x_2^*)u_2 + \cdots + (x_n - x_n^*)u_n = 0.$$

Example: Find the equation of the plane if the fixed point is $(1, -1, 2)$ and the vector is $u = [2, -1, 3]'$

Linear maps

Let A be a matrix of size 2×2 and we consider the map from $\mathbb{R}^2$ to $\mathbb{R}^2$,

$$x \mapsto Ax,$$

here $x = \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right]$.

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The map satisfies the following: for any $x, y \in \mathbb{R}^2$

1. $A(x + y) = A(x) + A(y)$
2. $A(\lambda x) = \lambda A(x)$, for any $\lambda \in \mathbb{R}$.

We say that such a map is linear.

Example: consider the linear map with $A = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$.

Rotation

The linear map with matrixes of the form

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

It rotates a vector by an angle θ, (counterclockwise if $\theta > 0$ or clockwise if $\theta < 0$).

Example: let $A = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$.

1) Find the image of the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

2) Find the inverse image of the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.