Advanced Mathematics 2 - Linear Algebra

Chapter 1: Systems of Linear Equations

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Topics

1. System of linear equations
2. Gaussian elimination
   - Elementary operations
   - Row reduction and echelon forms
   - Solutions of linear systems
3. Applications
   - Homogeneous system in Economics
   - Network Flow

Topics

Definitions

* A linear equation in the variables $x_1, x_2, \ldots, x_n$:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \quad (b, a_i \in \mathbb{R}\ \forall i)$$

* A system of m linear equations in n unknowns $x_1, x_2, \ldots, x_n$ is a set of equations of the form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = b_2 \\ \ldots\ldots\ldots\ldots\ldots & .. \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = b_m \end{cases} \qquad (*)$$

If $b_i = 0, \forall i$, the system is a homogeneous system.
Otherwise, the system is a non-homogeneous system.
* A solution of the system: a set of numbers $x_1, x_2, \ldots, x_n$ which satisfy all the m equations.
$\Rightarrow$ The solution set: the set of all possible solutions.

A system of linear equations has

1. no solution ($\Rightarrow$ the system is inconsistent), or
2. exactly one solution or infinitely many solutions ($\Rightarrow$ the system is consistent).

Example 1:

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases} \quad (1) \quad \longrightarrow A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$$

$\uparrow$

The coefficient matrix

$\Downarrow$

The augmented matrix

$$\overline{A} = [A|b] = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

$\Rightarrow$ **Matrix notation**: If m and n are positive integers, an $m \times n$ matrix is a rectangular array of numbers with m rows and n columns.

Matrix equation

The system $(*)$ can be written in the matrix form: $Ax = b$ with

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & \cdots & . \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \ x = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}, \text{ and } b = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_m \end{bmatrix}$$

$\uparrow$

the coefficient matrix

The augmented matrix: $\overline{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ . & . & \cdots & . & . \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$

Topics

1. System of linear equations

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3. Applications
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Elementary operations

The basic strategy: replacing one system with an equivalent system that is easier to solve.

Elementary operations for equations

1. **(Interchange)** Interchange two equations.
2. **(Scaling)** Multiply an equation by a nonzero constant.
3. **(Replacement)** Replace one equation by the sum of itself and a multiple of other equation.

Elementary row operations

1. **(Interchange)** Interchange two rows.
2. **(Scaling)** Multiply a row by a nonzero constant.
3. **(Replacement)** Replace one row by the sum of itself and a multiple of other row.

Echelon matrix

1. A **nonzero row or column** in a matrix.
2. A **leading entry of a row**

Echelon matrix

A matrix is in echelon form (or row echelon form) if:

1. All nonzero rows are above zero rows.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

Reduced echelon matrix

If a matrix in echelon form satisfies the following additional conditions then it is in reduced echelon form:

1. The leading entry in each nonzero row is 1.
2. Each leading 1 is the only nonzero entry in its column.

Row reduction algorithm

Pivot position in a matrix A: a location in A that corresponds to a leading 1 in the reduced echelon form of A.

Pivot column: a column of the matrix that contains a pivot position.

Example 2: Apply elementary row operations to transform the following matrix into echelon form and then into reduced echelon form:

$$B = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Solutions of linear systems

Whenever a system is consistent, applying the row reduction algorithm to *the augmented matrix* to obtain the solution set.

Basic variables: variables corresponding to pivot columns.

The other variables are called free variables.

Solutions of linear systems

Gaussian elimination

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, STOP; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system corresponding to the matrix obtained.
5. Express the basic variables in terms of free variables.

Example 3: Find the general solution of the following system:

$$\begin{cases} x_1 - 2x_2 - x_3 + 3x_4 = 0 \\ -2x_1 + 4x_2 + 5x_3 - 5x_4 = 3 \\ 3x_1 - 6x_2 - 6x_3 + 8x_4 = -3 \end{cases}$$

Topics

Homogeneous system in Economics

Example 1: Suppose an economy consists of the Coal, Electric (power), and Steel sectors, and the output of each sector is distributed among the various sectors as shown in Table 1, where the entries in a column represent the fractional parts of a sector's total output.

TABLE 1 A Simple Economy

Distribution of Output from:			
Coal	**Electric**	**Steel**	**Purchased by:**
.0	.4	.6	Coal
.6	.1	.2	Electric
.4	.5	.2	Steel

Network Flow

Example 2: The network in Figure 2 shows the traffic flow (in vehicles per hour) over several one-way streets in downtown Baltimore during a typical early afternoon. Determine the general flow pattern for the network.

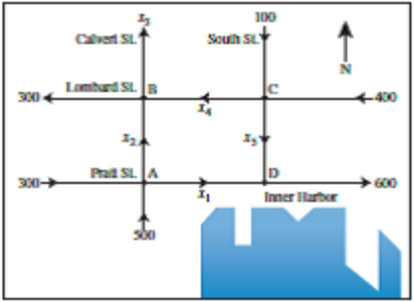

FIGURE 2 Baltimore streets.