Advanced Mathematics 2 - Linear Algebra

Chapter 2: Matrix Algebra - Determinant

Viet Nga PHAM

Department of Mathematics, Faculty of Information Technology,
Vietnam National University of Agriculture

Topics

Topics

Matrix: Definition

Definition

Let $m, n \in \mathbb{N}$. A rectangular array of real numbers with m rows, n columns is called a matrix of size $m \times n$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ . & . & \ldots\ldots & . \\ a_{i1} & a_{i2} & \ldots\ldots & a_{in} \\ . & . & \ldots\ldots & . \\ a_{m1} & a_{mj} & \ldots\ldots & a_{mn} \end{bmatrix}, \qquad A = [a_{ij}]_{m \times n}$$

- a_{ij}: the (i, j)-entry of A, lying in the ith row and jth column of A.
- Each column j of A is a list of m real numbers, which identifies a vector in $\mathbb{R}^m$, denoted by $\mathbf{a}_j$.
  $\longrightarrow$ The matrix A can be written as $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \ldots \mathbf{a}_n \end{bmatrix}$

Matrix: Definition

- A zero matrix
- A square $n \times n$ matrix
- The diagonal entries $\longrightarrow$ the main diagonal
- A diagonal matrix $\longrightarrow$ the $n \times n$ identity matrix I_n.
- Two matrices are equal if they have the same size and their corresponding entries are equal.

Sums and Scalar Multiples

If A and B are $m \times n$ matrices, r is a scalar, then

- The sum $A + B$ is the $m \times n$ matrix whose columns are the sums of the corresponding columns in A and B (i.e., each entry in $A + B$ is the sums of the corresponding entries in A and B).
- The scalar multiple rA is the matrix whose columns are r times the corresponding columns in A.
- $-A$ stands for $(-1)A$, so $A - B = A + (-1)B$.

$\longrightarrow$ Properties:...

Matrix Multiplication

If $A = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$ is an $1 \times n$ matrix and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ is an $n \times 1$ matrix then the product AB is the 1×1 whose the only entry is

$$c = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$$

Row-Column rule

If $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ is an $m \times n$ matrix and $B = \begin{bmatrix} b_{jk} \end{bmatrix}$ is an $n \times p$ matrix then the product AB is the $m \times p$ whose the (i, k)-entry is

$$c_{ik} = a_{i1} b_{1k} + a_{i2} b_{2k} + \ldots + a_{in} b_{nk}$$

Attention: The number of columns of A must match the number of rows in B.

Properties:

- Powers of a matrix
- The transpose of a matrix

Topics

Definition

A is an 1×1 matrix

The determinant of $A = \begin{bmatrix} a \end{bmatrix}$ is the scalar: $\det A = \begin{vmatrix} a \end{vmatrix} = a$

A is an 2×2 matrix

The determinant of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the scalar: $\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

A is an 3×3 matrix

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then the determinant of A is the scalar:

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

A is an $n \times n$ matrix, $n \geq 2$

Let $A = \left[a_{ij}\right]$ be an $n \times n$ matrix.
The determinant of A is the scalar:

$$\det A = a_{11}\det A_{11} - a_{12}\det A_{12} + \cdots + (-1)^{1+n} a_{1n}\det A_{1n}$$

where A_{ij} denote the $(n-1) \times (n-1)$ matrix obtained from A by deleting row i and column j.

Definition: (i,j)-cofactor

Given the $n \times n$ matrix A.
The (i,j)-cofactor $C_{ij}(A)$ is the scalar defined by

$$C_{ij}(A) = (-1)^{i+j} det(A_{ij})$$

Here $(-1)^{i+j}$ is called the sign of the (i,j)-position.

The Cofactor Expansion

The Cofactor Expansion along the first row

Let $A = \left[a_{ij}\right]$ be an $n \times n$ matrix. Then

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

Cofactor Expansion Theorem

The determinant of an $n \times n$ matrix A can be computed by using the cofactor expansion along any row or column of A.

The expansion across the i^{th} row:

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

The cofactor expansion down the j^{th} column:

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Properties of Determinants

Theorem 1

If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

The secret of determinants lies in **how they change** when **row operations** are performed !

Theorem 2

Let A be a square matrix.

1. If two rows of A are interchanged to produce B, then
$$\det B = -\det A$$
2. If one row of A is multiplied by α to produce B, then
$$\det B = \alpha.\det A$$
3. If a multiple of one row of A is added to another row to produce a matrix B, then
$$\det B = \det A$$

Properties of Determinants

Theorem 3

If A is a square matrix, then $\det A^T = \det A$

(For Proof: Using The Principle of Mathematical Induction)

Theorem 4 (Multiplicative Property)

If A and B are $n \times n$ matrices, then $\det AB = (\det A).(\det B)$

Topics

Definition

Definition

An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix B such that

$$AB = I \text{ and } BA = I$$

where $I = I_n$, the $n \times n$ identity matrix.
In this case, B is an **inverse** of A.

Example: $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ and $B = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$

Attention: In fact, B is uniquely determined by A. This unique inverse is denoted by A^{-1}.

A singular matrix: a matrix that is not invertible.

An invertible matrix is called a nonsingular matrix.

Properties

1

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$ then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$ then A is not invertible.

The quantity $ad - bc$ is called the **determinant** of A.

2

If A is an invertible $n \times n$ matrix then for each $\mathbf{b}$ in $\mathbb{R}^n$, the system $Ax = b$ has the unique solution $x = A^{-1}\mathbf{b}$

The Invertible Matrix Theorem

Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent.

1. A is an invertible matrix.
2. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
3. The equation $A\mathbf{x} = \mathbf{b}$ has the unique solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
4. There is an $n \times n$ matrix C such that $CA = I$.
5. There is an $n \times n$ matrix D such that $AD = I$.
6. A^T is an invertible matrix.

Cramer's Rule

Theorem

Let A be an invertible $n \times n$ matrix. For any b in $\mathbb{R}^n$, the unique solution x of $Ax = b$ has entries given by

$$x_j = \frac{\det A_j(b)}{\det A}, \qquad \forall j = 1, 2, ..., n$$

where $A_j(b)$ be the matrix obtained from A by replacing column j by the vector b.

An Inverse Formula

Cramer's rule leads easily to a general formula for the inverse of an $n \times n$ matrix A.

Theorem

Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

$\operatorname{adj} A$ is the **adjugate** of A.

$$\operatorname{adj} A = \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}$$

Topics

Model

Suppose a nation's economy is divided into n sectors that produce goods or services.

x: **production vector** in $\mathbb{R}^n$ that lists the output of each sector for one year.

Suppose another part of the economy does not produce goods or services but only consumes them.

d: **final demand vector** that lists the values of goods and services demanded from the various sectors by the nonproductive part of the economy. **Q?** Is there a production level **x** such that the amounts produced will exactly balance the total demand for that production? that means,

$$\begin{pmatrix} \textit{amount} \\ \textit{produced} \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} \textit{intermediate} \\ \textit{demand} \end{pmatrix} + \begin{pmatrix} \textit{final} \\ \textit{demand} \\ \mathbf{d} \end{pmatrix}$$

where, the **intermediate demand** lists the goods that the various sectors need as inputs for their own productions.

Model

Basic assumption: for each sector, there is **a unit consumption vector** in $\mathbb{R}^n$ that lists the inputs needed per unit of output of the sector.
(All input and output are measured in millions of dollars)
Example 1: c_1, c_2, c_3 are unit consumption vectors of Manufacturing, Agriculture, and Services. What amounts will be consumed by the manufacturing sector if it decides to produce 100 units?
The consumption matrix $\begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}$, named C, is

$$C = \begin{bmatrix} .50 & .40 & .20 \\ .20 & .30 & .10 \\ .10 & .10 & .30 \end{bmatrix}$$

The Leontief Input-Output Model

$$\mathbf{x} = C\mathbf{x} + \mathbf{d}$$

? Solution ?
Example 2: p.133