Chapter 3. Mathematics of Finance

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Content

Revision: percentage, scale factor, percentage change

$$\text{Scale factor } = \frac{\text{new data}}{\text{old data}}$$

$$\text{Percentage change } = \left| \frac{\text{new data} - \text{ old data}}{\text{old data}} \right|$$

$$\text{Percentage change } = \text{ Scale factor } - 1$$

Examples: page 198, 200, 201, 202

ppp 199, 201, 202, 203

Index numbers

Example: (page 204) Table 3.1 shows the values of household spending (in billions of dollars) during a 5-year period. Calculate the index numbers when 2011 is taken as the base year and give a brief interpretation.

	Year				
	2010	2011	2012	2013	2014
Household spending	686.9	697.2	723.7	716.6	734.5

$$\text{index number} = \text{scale factor from base time} \quad \times 100$$

Note that the index number of 2011 (base year) is 100.
For example, the index number of 2012 is

$$\frac{723.7}{697.2} \times 100 = 103.8$$

Result:

	Year				
	2010	2011	2012	2013	2014
Household spending	686.9	697.2	723.7	716.6	734.5
Index number	98.5	100	103.8	102.8	105.3

Example: page 205 and find the percentage changes.

* ppp 206

Inflation: nominal data and real data (page 207)

The annual rate of inflation is the average percentage change in a given selection of goods and services (called consumer basket), over the previous year.

Nominal data are prevailed at the time.

Real data are the values that have been adjusted to take inflation into account (with a chosen base year).

Example: (page 207) Table 3.8 shows the price (in thousands of dollars) of an average house in a certain town during a 5-year period. The price quoted is the value of the house at the end of each year. Use the annual rates of inflation given in Table 3.9 to adjust the prices to those prevailing at the end of 1991. Compare the rise in both the nominal and real values of house prices during this period.

	Year				
	1990	1991	1992	1993	1994
Average house price	72	89	93	100	106
Annual rate of inflation		10.70%	7.10%	3.50%	2.30%

* ppp 209

*Key terms: page 209

*HW: Exercise 3.1, page 209-213

Simple interest and compound interest, page 215

Example, (page 215): Compare interests and the investments generated at the end of each year if an amount of \$500 is invested for 5 years at 10 % interest simple and compounded annually.

We call:
- The principal, denoted by P, the original sum of money;
- The future value, denoted by S, the final sum of money.

Let r % be the interest rate compounded annually. Then after $t = n$ year,

$$S = Pq^n, \text{ for } q = 1 + r\%$$

We note that the scale factor after each year is

$$q = q = 1 + r\%$$

Example: find the value, in 4 years' time, of \$10 000 invested at 5% interest compounded annually.
Example: page 217

*ppp 217, 219

Discrete compounding:
Let r % be the interest rate compounded periodically, and $t = n$ is the number of time periods. We have also

$$S = Pq^n, \text{ for } q = 1 + r\%$$

Note that the interest rate might be compounded either annually, or semi-annually, or quarterly, or monthly, or weekly.

For example if the interest rate is 8 % compounded quarterly, the during time is $t = 5$ then $r\% = 8\%/4 = 2\%$, and $n = 5 \times 4 = 20$.

Example: page 219

Continuous compounding

If the interest rate r % is compounded continuously, then after t year,

$$S = Pq^t, \text{ for } q = e^{r\%}$$

Example: (page 221) A principal of \$2000 is invested at 10% interest compounded continuously. After how many days will the investment first exceed \$2100?

*ppp 222

Annual percentage rate- APR

The APR (or AER) is the rate of interest which, when compounded annually for loans (or savings), produces the same yield as the nominal rate of interest.

Example: (page 222) Determine the annual equivalent rate of interest of a deposit account that has a nominal rate of 6.6 % compounded monthly.

*ppp 223
*HW: Exercise 3.2, page 225-, Problems 1-15

Geometric progression

Ex: 1, 2, 4, 8, 16, 32,...
Ex: $a, aq, aq^2, aq^3, ...$ (q is the geometric ratio)

Geometric series: for $a \in \mathbb{R}$ and $q \neq 1$, the sum of the n first terms of a geometric progression is

$$a + aq + aq^2 + ... + aq^{n-1} = a\left(\frac{q^n - 1}{q - 1}\right) = \frac{u_{n+1} - u_1}{q - 1}$$

Example: evaluate the geometric series

$$500(1.1) + 500(1.1)^2 + 500(1.1)^3 + ... + 500(1.1)^{25}$$

* ppp 231

Applications of geometric series: savings and loans

Savings: discussion about sinking fund
Send equal amounts to save into a bank account periodically (ex: at the same time each year or month) and assume that the interest rate does not change.

Suppose that an equal amount a is saved at the beginning at each period, and the constant interest rate is $r\%$, then total savings after n periods is

$$aq + aq^2 + \ldots + aq^n = aq\left(\frac{q^n - 1}{q - 1}\right),$$

for $q := 1 + r\%$.

Example: (page 232) A person saves \$100 in a bank account at the beginning of each month. The bank offers a return of 12% compounded monthly.

(a) Determine the total amount saved after 12 months.

(b) After how many months does the amount saved first exceed \$2000?

*ppp 233

Loans: discussion about repayment for a loan by installments

Example: A person borrows a loan of \$1000 at 8% interest compounded monthly. If the person repays an amount of \$100 at the end of each month. Find the bedt owed (the outstanding debt) at the end of 5-th month.

Let the original loan L, equal periodically installments a, the interest rate $r\%$.
Let $q := 1 + r\%$. The debt owed at the end of $n-$th period is

$$Lq^n - \left(a + aq + aq^2 + \ldots + aq^{n-1}\right)$$
$$= Lq^n - a\left(\frac{q^n - 1}{q - 1}\right)$$

The debt is paid off if the periodically repayments are

$$a = Lq^n \div \frac{q^n - 1}{q - 1}$$

Example: (page 232) Determine the monthly repayments needed to repay a \$100 000 loan which is paid back over 25 years when the interest rate is 8% compounded annually.

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$L = \$100000$, $r\% = 8\% \Rightarrow q = 1.08$, $n = 25$

End of year	Outstanding debt
1	98632.08
2	97154.73
3	95559.18
5	91975.20
10	80184.14
20	37403.21
25	0

*ppp 235, 236

Key terms: page 237

*HW: Exercise 3.3, page 237-
HW: Exercise 3.3*, page 238-239, problems 2, 4, 5, 8

Formal mathematics: pages 256-

Discounting: find the present value for a future value under discrete compounding or continuous compounding:

$$P = Sq^{-t}, \text{ for } q = 1 + r\%$$

or

$$P = Sq^{-t}, \text{ for } q = e^{r\%}$$

Example: (page 241) Find the present value of \$1000 in 4 years's time if the discount rate is 10% compounded
(a) semi-annually
(b) continuously

*ppp 241

Appraising an investment project by using the Net present value and the Initial rate of return

Net present value NPV, page 241

$$NPV = \text{Present value of revenue, } P - \text{Present value of costs, } P_0$$

If $NPV > 0$ then the project is profitable

Initial rate of return, page 242

This is an annual rate which, when applied to the initial outlay, yields the same return as the project after the same number of years. $(P_0 \Rightarrow S)$

The investment is considered worthwhile provided the IRR exceeds the market rate.

Example: (page 242) A project requiring an initial outlay of \$15 000 is guaranteed to produce a return of \$20 000 in 3 years? time. Use the

(a) net present value

(b) internal rate of return

methods to decide whether this investment is worthwhile if the prevailing market rate is 5% compounded annually. Would your decision be affected if the interest rate were 12%?

*ppp 243

Example page 244, and pp 245

Calculating the present value of an annuity , page 245

An annuity is a sequence of regular equal payments

Present value of annuity (payments are made at the end of time periods) is

$$Sq^{-1} + Sq^{-2} + \dots + Sq^{-n} = \frac{Sq^{-n-1} - Sq^{-1}}{q^{-1} - 1},$$

for $q := 1 + r\%$.

Example: (page 245) Suppose that the interest rate is 7% compounded annually. Find the present value of an annuity which give a regular income of \$10 000 at the end of each year

(a) for the next 10 years;
(b) for many years (forever).

*ppp 246

Cash flow: $S = \$10000$, $r\% = 7\%$, $q = 1.07$, $P = S \times 1.07^{-t}$

t	present value
1	9345.794393
2	8734.387283
3	8162.978769
4	7628.952120
5	7129.861795
6	6663.422238
7	6227.497419
8	5820.091046
9	5439.337426
10	5083.492921
Present value of annuity	70235.81541

Revenue flow: example page 247

Initial outlay $P_0 = \$20000$, $r\% = 7\%$

	Revenue		Discounted revenue	
t	Project A	Project B	Project A	Project B
1	6000	10000	5405.405	9009.009
2	3000	6000	2434.867	4869.734
3	10000	9000	7311.913	6580.722
4	8000	1000	5269.847	658.730
Total	27000	26000	20422.034	21118.19

NPV of Proj. A $= \$422.034 <$ NPV of Proj. B $= \$1118.19$

*ppp 246

Computing IRR of an investment project, example and ppp 248-250

a) Initial outlay $P_0 = \$20000$, returns: \$ 8000 and \$ 15000 at the and of 1st year and 2nd year.

Computing IRR of an investment project, example and ppp 248-250

a) Initial outlay $P_0 = \$20000$, returns: \$ 8000 and \$ 15000 at the and of 1st year and 2nd year.

Let $q := 1 + r\%$, the sum of present values must equal the initial investment of \$20 000,

$$8000q^{-1} + 15000q^{-2} = 20000$$

$$8000q + 15000 = 20000q^2$$

$$20q^2 - 8q - 15 = 0$$

$$q = 1.089 \text{ or } q = -0.69 \text{ (remouved)}$$

Then $r\% = 8.9\%$

*Give comments if the prevailing market rate is 10 % compounded annually?

b) Initial outlay $P_0 = \$5000$, returns: \$ 1000, \$ 2000 and \$ 3000 at the and of 1st year, 2nd year and 3rd year.

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$$1000q^{-1} + 2000q^{-2} + 3000q^{-3} = 5000$$

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$$1000q^{-1} + 2000q^{-2} + 3000q^{-3} = 5000$$

Use the trial and error method:

r %	**5%**	**6%**	**7%**	**8%**	**9%**	**10%**
q	1.05	1.06	1.07	1.08	1.09	1.1
value	1308	5242	5130	5022	4917	4816

This table indicates that $r\%$ is between 8% and 9%.
For more accurate, we simply try further values between 8% and 9%. For example, check $r\% = 8.5\%$ gives 4969...

Calculating present values of government bonds,
example and ppp 251-252

Example page 251: A 10-year bond is originally offered by the government at \$5000 with an annual return of 9%. Assuming that the bond has 4 years left before redemption, calculate its present value assuming that the prevailing interest rate is
(a) 5%, (b) 7%, (c) 9%, (d) 11%, (e) 13%

Calculating present values of government bonds, example and ppp 251-252

Example page 251: A 10-year bond is originally offered by the government at \$5000 with an annual return of 9%. Assuming that the bond has 4 years left before redemption, calculate its present value assuming that the prevailing interest rate is (a) 5%, (b) 7%, (c) 9%, (d) 11%, (e) 13%

		Present value				
End of year	Cash flow	5%	7%	9%	11%	13%
1	450	429	421	413	405	398
2	450	408	393	379	365	352
3	450	389	367	347	329	312
4	5450	4484	4158	3861	3590	3343
Total present value		5709	5339	5000	4690	4405

* Discussion on speculative demand and interest rate, page 251.

Key terms: page 252

HW: Exercise 3.4*, page 253-
HW: Exercise 3.4*, page 87-88, problems 2, 4, 6, 8, 10

Formal mathematics: pages 256-

THANK YOU VERY MUCH!