Chapter 4. Differentiation

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Content

Motivation and definition

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Average/ Instantaneous Rate of Change

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Examples:
- The inst. rate of growth
- Velocity; Acceleration: the rate of change of velocity with respect to time
- The rate of a chemical reaction
- The marginal revenue: the rate of change of total revenue...

The slope (or gradient) of a straight line

$$\text{slope } = \frac{\Delta y}{\Delta x}$$

Example: page 261

The derivative as an Instantaneous Rate of Change

Let a function $y = f(x)$. For $h \in \mathbb{R}$, we denote

$$\Delta x = (x+h) - x = h, \ \Delta y = f(x+h) - f(x)$$

Definition

The derivative of a function f at x, denoted by $f'(x)$, is

$$f'(x) = \lim_{h \to 0} \frac{\Delta y}{\Delta x},$$

provided that the limit exists. Leibniz notation:

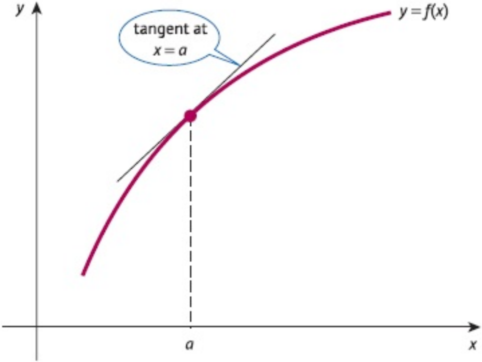
$$f'(x) = \frac{df}{dx}$$

Remark: for $\Delta x = h$ small, then $f'(x) \simeq \frac{\Delta y}{\Delta x}$
Example: Find the difference quotient at $x = 1$ and then $f'(1)$ for the functions $f(x) = x^2$.

Geometric interpretation of the derivative

Proposition

If f has a derivative at $x = a$, then $f'(a)$ is the slope of the tangent line at the point $(a, f(a))$ to the graph of f.



Example: find the tangent line to $f(x) = x^2$ at $x = 2$.

Differentiation of fundamental functions

- $(x^\alpha)' = \alpha(x^{\alpha-1})$, for any $\alpha \in \mathbb{R}$;
  For example: $\sqrt{x}' = \frac{1}{2\sqrt{x}}$.
- $(e^x)' = e^x$;
  $(a^x)' = (\ln a)a^x$, for $0 < a \neq 1$;
- $(\ln x)' = \frac{1}{x}$;
  $(\log_a x)' = \frac{1}{x \ln a}$.

Example...

Basic rules of differentiation

In the following, $u = u(x), v = v(x)$ are differentiable functions of x:

$$(u+v)' = u' + v',$$

$$(ku)' = ku', k \in \mathbb{R},$$

$$(uv)' = u'v + uv',$$

$$(\frac{u}{v})' = \frac{u'v - uv'}{v^2}.$$

Example

The chain Rule

Example 1: find the derivative of $f(x) = (x^3 - 2x - 1)^5$.

Chain rule

$$(f \circ u)'(x_0) = f'(u_0)u'(x_0),$$

where $u_0 = u(x_0)$.

This above expression can be written in Leibniz notation as

$$\frac{d}{dx}[f \circ u(x)] = \frac{df}{du}\frac{du}{dx}.$$

Example 2: find the derivative of the function

$$h(x) = \ln(x^2 + 1).$$

Differentiation of fundamental functions

In the following, $u = u(x)$ is a differentiable function of x.

- $(u^\alpha)' = \alpha(u^{\alpha-1})u'$, for any $\alpha \in \mathbb{R}$;
- $(e^u)' = e^u u'$;
  $(a^u)' = (\ln a)a^u u'$, for $0 < a \neq 1$;
- $(\ln u)' = \frac{u'}{u}$;
  $(\log_a u)' = \frac{u'}{(\ln a)u}$.

Higher derivatives

Let f be a function. The f' is called **the first order derivative**

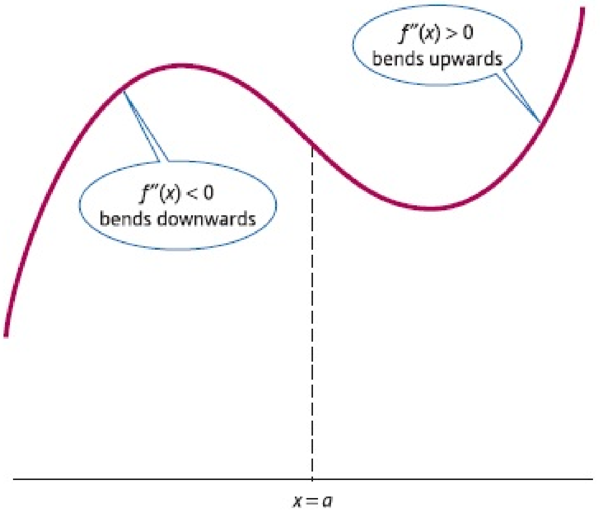
The second derivative:

$$f''(x) = \frac{d^2y}{dx^2} = [f'(x)]' = \frac{d}{dx}(\frac{dy}{dx})$$

The same way for **the third derivative**, denoted f''',

Example: let $f(x) = \sqrt{x}, x \geq 0$.

Convexity and concavity: page 278



HW: Exercise 4.1, page 270, problems 5, 6

HW: Exercise 4.1, page 271, problem 2

HW: Exercise 4.2, page 280, problems 2, 8

HW: Exercise 4.2*, page 280, problems 1, 2

HW: Exercise 4.4, page 305-, problems 2, 6, 7

HW: Exercise 4.4, page 305-, 1, 2, 3, 6, 7, 8

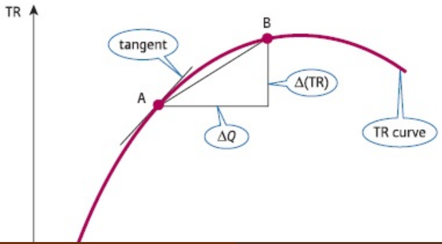
Let TR be the total revenue of a good. The marginal revenue MR is defined by

$$MR = \frac{d(TR)}{dQ}$$

For ΔQ small enough (for example $\Delta Q = 1$),

$$MR \simeq \frac{\Delta(TR)}{\Delta Q},$$

$$\Delta(TR) \simeq MR \times \Delta Q$$



Example: If the demand function is

$$P = 120 - 3Q$$

find an expression for TR in terms of Q.
Find the value of MR at $Q = 10$ using

(a) differentiation.
(b) the 1 unit increase approach.

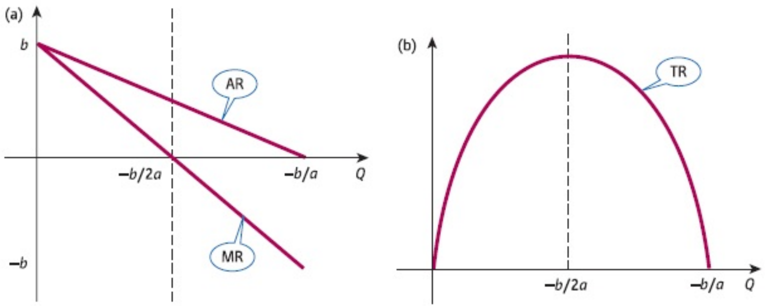
*ppp 285, 286

The relationship between MR and Average revenue

Monopoly: $P = aQ + b$, $TR = PQ = aQ^2 + bQ$,
$MR = 2aQ + b$.

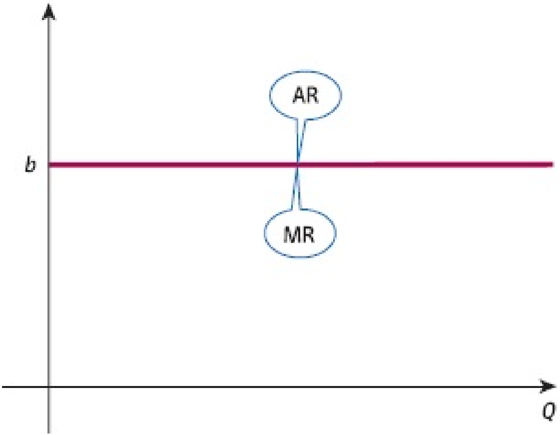
The average revenue AR is defined by

$$AR = \frac{TR}{Q} = P$$



Perfect competition: page 287

$$P = b, TR = PQ = bQ, MR = b = P, AR = \frac{TR}{Q} = P = b$$



Marginal cost: page 287
Let TC be the total cost of a good. The marginal cost is defined by

$$MC = \frac{d(TC)}{dQ}$$

For ΔQ small enough, $MC \simeq \frac{\Delta(TC)}{\Delta Q}$, and so,

$$\Delta(TC) \simeq MC \times \Delta Q$$

Example: (page 289) If the average cost function of a good is

$$AC = 2Q + 6 + \frac{13}{Q}$$

find an expression for MC. If the current output is 15, estimate the effect on TC of a 3 unit decrease in Q.

*ppp 289

Marginal product of labour: page 290
Let Q be the Production function of a good, which is assumed to be a function of labour L and capital K: $Q = Q(L, K)$.
We define the marginal product of labour MP_L by

$$MP_L = \frac{d(Q)}{dL}$$

Example: (page 290) If the production function is

$$Q = 300\sqrt{L} - 4L,$$

calculus the value of MP_L when

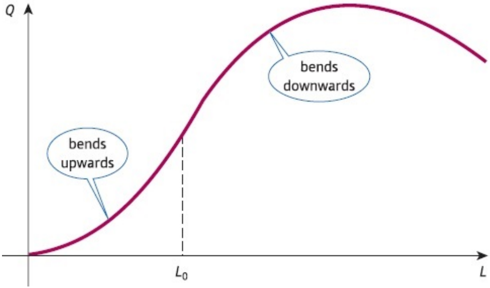
$$L = 1; \quad L = 9; \quad L = 100; \quad L = 2500,$$

and discussion the implication of these results.

*ppp 289

Law of diminishing marginal productivity / returns: page 291

The increase in output due to a 1 unit increase in labor will eventually decline, i.e. once the size of workforce has reached a certain threshold level, the MP_L will get smaller.



$$\frac{d(MP_L)}{dL} = \frac{d^2(Q)}{dL^2} < 0, \text{ for } L > L_0$$

*ppp 292

Consumption and savings

We suppose that

$$Y = C + S$$

Then,

$$\frac{Y}{dY} = \frac{C}{dY} + \frac{S}{dY} = MPC + MPS = 1$$

Example: If the consumption function is

$$C = 0.01Y^2 + 0.2Y + 50$$

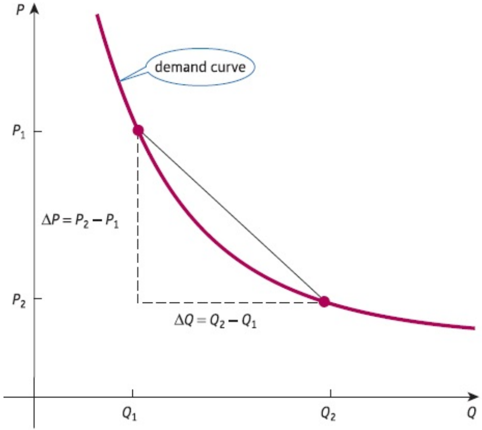
calculate MPC and MPS when $Y = 30$.

*ppp 294

Key term: page 294

*HW: Exercise 4.3, page 294-295
HW: Exercise 4.3, page 295-296, Problems 1, 2, 3, 5, 6

Price elasticity of demand



The price elasticity of demand is defined by

$$E = \frac{\text{percentage change in demand}}{\text{percentage change in price}} = \left| \frac{\Delta Q/Q}{\Delta P/P} \right|$$

$$E = -\frac{P}{Q} \times \frac{\Delta Q}{\Delta P}$$

Demand is said to be inelastic if $E < 1$, unit elastic if $E = 1$ and elastic if $E > 1$.

Arc elasticity: a section of the demand curve between two points (Q_1, P_1) and (Q_2, P_2). Then we take,

$$P = \frac{P_1 + P_2}{2}, \;\; Q = \frac{Q_1 + Q_2}{2}$$

Example: Determine the elasticity of demand when the price falls from 136 to 119, given the demand function

$$P = 200 - Q^2$$

*ppp 310

Point elasticity

$$E = -\frac{P}{Q} \times \frac{dQ}{dP}$$

Example: (page 310) given the demand function

$$P = 50 - 2Q$$

find the elasticity when the price is 30. Is the demand inelastic, unit elastic or elastic at this price?

*ppp 311

The derivative of an inverse function

$$Q = (P) \Rightarrow P = f(Q),$$

then

$$\frac{dQ}{dP} = \frac{1}{\frac{dP}{dQ}}$$

Example: page 312 and ppp 313

*The relationship between price elasticity of demand and marginal revenue

$$MR = P\left(1 - \frac{1}{E}\right)$$

Price elasticity of supply

It is defined in an analogous way to that of demand.

$$E = \frac{\text{percentage change in supply}}{\text{percentage change in price}}$$

Here

$$E = \frac{\Delta Q/Q}{\Delta P/P} = \frac{P}{Q} \times \frac{\Delta Q}{\Delta P}$$

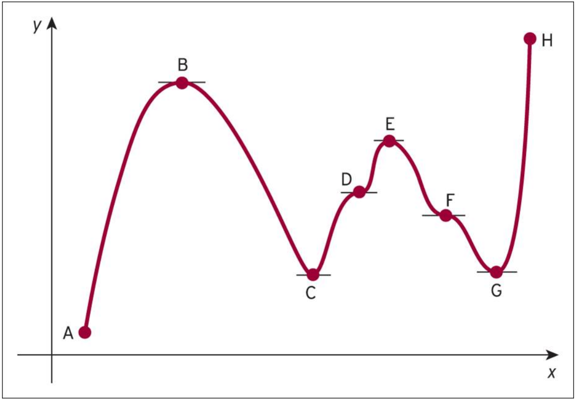
Example: page 314 and ppp 315

Key term: page 319

*HW: Exercise 4.5, page 319-320, Problems 3, 4, 5, 7

HW: Exercise 4.5, page 320-321, Problems 3, 4, 6, 7

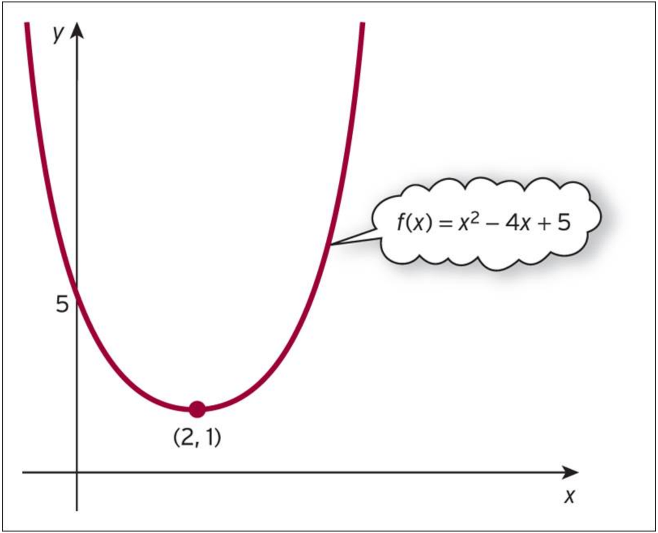
Stationary (critical, turning) points, local (relative) maximum & minimum points (extrema)



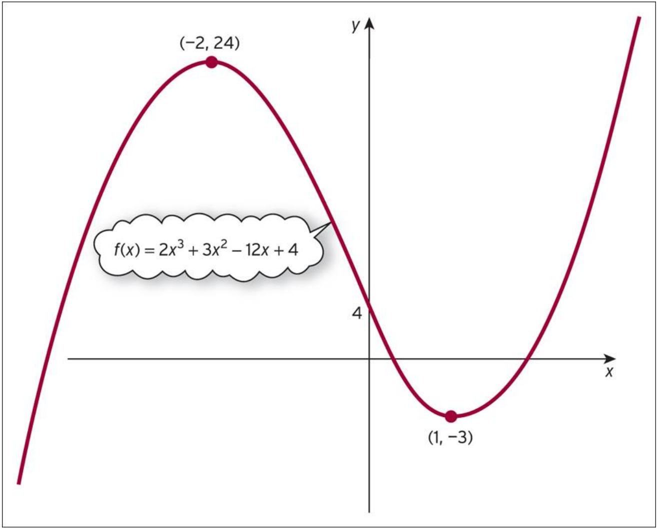
Classification of stationary points

Stationary point	$f'(a) = 0$		
	Sign of f'		Sign of f''
Minimum point	$f'(a) = 0$,	$f'(x) < 0$ for $x < a$ $f'(x) > 0$ for $x > a$	$f''(a) > 0$
Maximum point	$f'(a) = 0$,	$f'(x) > 0$ for $x < a$ $f'(x) < 0$ for $x > a$	$f''(a) < 0$
Reflection point	$f'(a) = 0$,	$f'(x) < 0$ for $x < a$ and $x > a$ or $f'(x) > 0$ for $x < a$ and $x > a$	$f''(x) < 0$ for $x < a$ $f''(x) > 0$ for $x > a$ or $f''(x) > 0$ for $x < a$ $f''(x) < 0$ for $x > a$

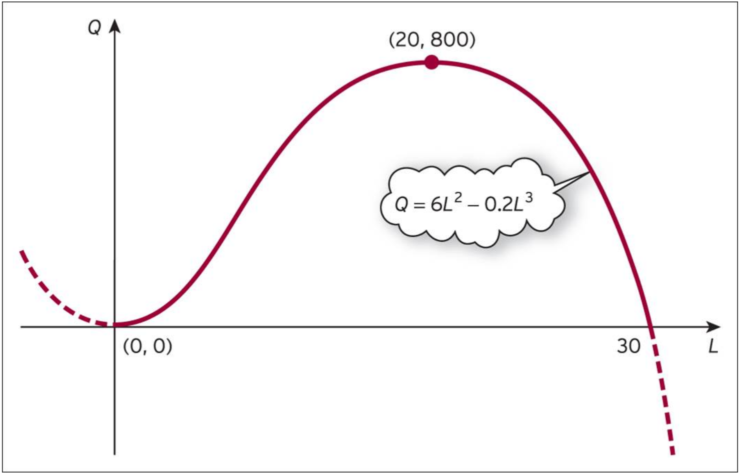
Example: page 324



Example: page 324



Example page 328: Maximizing output Q and average product of labor (or called also labor productivity) AP_L



AP_L max then $AP_L = MP_L$

Maximizing the government's total tax revenue: example page 336 and ppp 337

Maximizing total revenue and profit functions: example page 331 and ppp 333

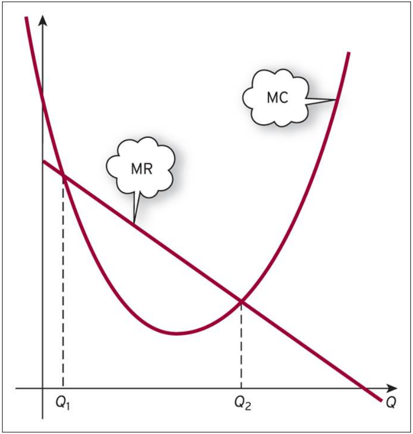
Minimizing the average cost function : example page 334 and ppp 335

Key terms: page 338

*HW: Exercise 4.6, page 338-339: problems 2, 5, 6, 8
HW: Exercise 4.6, page 340: problems 3, 6

Maximizing profit functions: ppp 343

$$\pi \text{ max then } MR = MC$$



Maximizing profit functions: for several markets with/ or without price discrimination

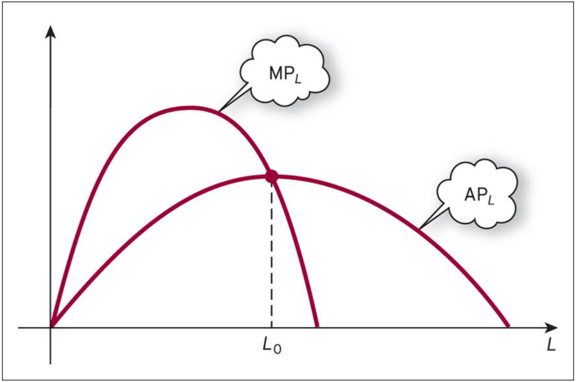
Example page 343, ppp 347

Remark: with a common marginal cost, profit maximization problem with price discrimination leads to

$$MR_1 = MR_2 \Leftrightarrow P_1\left(1 - \frac{1}{E_1}\right) = P_2\left(1 - \frac{1}{E_2}\right)$$

Maximizing average product of labor AP_L: page 348

$$AP_L = MP_L$$



Key terms: page 338

HW: Exercise 4.7, page 352: problems 1, 2, 3, 5

THANK YOU VERY MUCH!