Chapter 5.P2. Partial Differentiation

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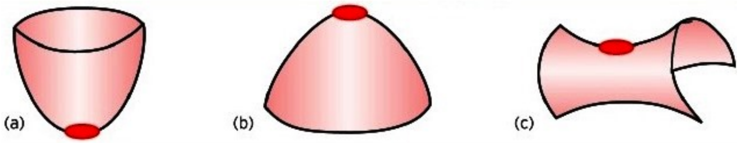
Content

Stationary points of a function of two variables $z = f(x, y)$:



Stationary points satisfy

$$f_x(x_0, y_0) = 0, \;\; f_y(x_0, y_0) = 0.$$

$$A = f_{xx}(x_0, y_0), B = f_{xy}(x_0, y_0), C = f_{yy}(x_0, y_0).$$

$$\triangle = AC - B^2 \,\square\, 0.$$

1. If $\triangle < 0$ then (x_0, y_0) is saddle point (c).
2. If $\triangle > 0$, and $A > 0$ then (x_0, y_0) is minimum point (a).
3. If $\triangle > 0$, and $A < 0$ then (x_0, y_0) is maximum point (b).

Example (page 417): Find and classify stationary points of the function $f(x,y) = x^3 - 3x + xy^2$.

$$f_x = 3x^2 - 3 + y^2, f_y = 2xy$$

$$A = f_{xx} = 6x, \;\; B = f_{xy} = 2y, \;\; C = f_{yy} = 2x$$

Step 1: find stationary points by solving simultaneous equations

$$\begin{cases} f_x = & 3x^2 - 3 + y^2 & = 0 \\ f_y = & 2xy & = 0 \end{cases}$$

Stationary points: $M_1(0, \sqrt{3})$, $M_2(0, -\sqrt{3})$, $M_3(1, 0)$, $M_4(-1, 0)$

Step 2: $\triangle = AC - B^2 \;\square\; 0$ at the stationary points

Stationary points	A	B	C	$\triangle$	Conclusion
$M_1(0, \sqrt{3})$					
$M_2(0, -\sqrt{3})$					
$M_3(1, 0)$					
$M_4(-1, 0)$					

Maximization of profit of a firm producing two goods

Example (page 419): Find the maximum profit and the values of Q_1 and Q_1 at which this is achieved.

$$TC = 2Q_1^2 + 2Q_1Q_2 + Q_2^2,$$

$$P_1 = 1000, \ P_2 = 800$$

The profit function is $\pi = TR - TC = TR_1 + TR_2 - TC$,

$$\pi = 1000Q_1 + 800Q_2 - (2Q_1^2 + 2Q_1Q_2 + Q_2^2)$$

Step 1: find stationary points by solving simultaneous equations

$$\begin{cases} \pi_{Q_1} = & 1000 - 4Q_1 - 2Q_2 & = 0 \\ \pi_{Q_2} = & 800 - 2Q_1 - 2Q_2 & = 0 \end{cases} \Leftrightarrow \begin{cases} Q_1 = & 100 \\ Q_2 = & 300 \end{cases}$$

The stationary point is $M(100, 300)$

Step 2: at the stationary point $M(100, 300)$:
$A = -4 < 0,\ B = -2,\ C = -2,\ \triangle = AC - B^2 = 4 > 0$
Therefore the profit is maximized if $Q_1 = 100, Q_2 = 300$, and the corresponding maximum profit is

$$\pi(100, 300) = \$170000$$

*ppp 421

Maximization of profit of a firm that sells a good in two markets with price discrimination

Example (page 422): Find the maximum profit:

$$P_1 + Q_1 = 500,$$

$$2P_2 + 3Q_2 = 720,$$

$$TC = 50000 + 20Q, \text{ where } Q = Q_1 + Q_2.$$

Maximization of profit of a firm that sells a good in two markets with price discrimination

Example (page 422): Find the maximum profit:

$$P_1 + Q_1 = 500,$$

$$2P_2 + 3Q_2 = 720,$$

$$TC = 50000 + 20Q, \text{ where } Q = Q_1 + Q_2.$$

We have $P_1 = 500 - Q_1$ and $P_2 = 360 - \frac{3}{2}Q_2$.
The profit function is $\pi = TR - TC = TR_1 + TR_2 - TC$,

$$\pi = P_1Q_1 + P_2Q_2 - (50000 + 20Q),$$

$$\pi = 480Q_1 - Q_1^2 + 340Q_2 - \frac{3}{2}Q_2^2 - 50000$$

Result: π has a maximum at $Q_1 = 240$ and $Q_2 = \frac{340}{3}$. Then the corresponding prices are $P_1 = \$260$, $P_2 = \$190$, and the maximum profit is $\pi = \$26866.67$

Key terms: page 425

*HW: Exercise 5.4, pages 425-426, problems 1, 3, 4

HW: Exercise 5.4, pages 426-427, problems 2, 3, 4, 5

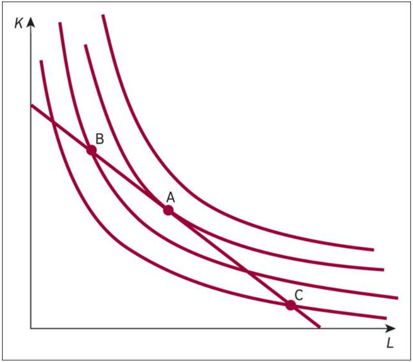
Optimize (Max/Min) a function $z = f(x, y)$, called the objective function subject to a constraint

$$\varphi(x, y) = M$$

Maximize Production function $Q = Q(K, L)$ subject to

$$P_K K + P_L L = M \Leftrightarrow K = -\frac{P_L}{P_K} L + \frac{M}{P_K} \text{ (isocost curve)},$$

where P_K and P_L are the costs of each unit of capital and labour respectively, M is a fixed amount.



The Q has a maximum Q_0 if the isocost line is tangential to the isoquant curve $Q = Q_0$. We have already shown in Section 5.2 that

$$MRTS = -\frac{dK}{dL} = \frac{Q_L}{Q_K} = \frac{MP_L}{MP_K}$$

So that at the maximum point:

$$\frac{MP_L}{MP_K} = \frac{P_L}{P_K} \Leftrightarrow \frac{MP_L}{P_L} = \frac{MP_K}{P_K}$$

Maximize Utility function $U = U(x_1, x_2)$ subject to the budgetary constraint:

$$P_1x_1 + P_2x_2 = M \Leftrightarrow x_2 = -\frac{P_1}{P_2}x_1 + \frac{M}{P_2} \text{ (isobudget line)},$$

where P_1 and P_2 are the costs of each unit of Good 1 and Good 2 respectively, M is a fixed budget for these goods.

At the maximum point, namely A, we have

$$\frac{P_1}{P_2} = \frac{U_{x_1}}{U_{x_2}} \Leftrightarrow \frac{U_{x_1}}{P_1} = \frac{U_{x_2}}{P_2}.$$

Remind as we have already shown in Section 5.2 that

$$MRCS = -\frac{dx_2}{dx_1} = \frac{U_{x_1}}{U_{x_2}}$$

Solve constrained optimization problem using the substitution method

Example (page 431): minimize the objective function $z = -2x^2 + y^2$ subject to

$$2x - y = 1.$$

*ppp 433

Example (page 433): maximize the production function $Q = 4LK + L^2$ subject to

$$K + 2L = 105$$

*ppp 435

Key terms: page 437

*HW: Exercise 5.5, pages 438-, problems 2, 4, 5, 6, 7, 8

HW: Exercise 5.5, pages 439-, problems 1, 2, 3, 4

Optimize (Max/Min) a function $z = f(x, y)$ s.t

$$\varphi(x, y) = M$$

Step 1: Introduce the Lagrange function

$$g(x, y) = f(x, y) - \lambda[\varphi(x, y) - M],$$

where the scale λ is called the Lagrange multiplier
Step 2: Find stationary points of g,

$$g_x = 0, g_y = 0, g_\lambda = 0$$

Step 3: Give optimal solutions

Example (page 444): Find the maximum profit,

$$P_1 = 50 - Q_1 + Q_2,\ P_2 = 30 + 2Q_1 - Q_2,$$

$$TC = 10Q_1 + Q_1Q_2 + 10Q_2,\ Q_1 + Q_2 = M = 15.$$

Then estimate the new optimal profit if $Q = M$ rises by 1 unit.

Example (page 444): Find the maximum profit,

$$P_1 = 50 - Q_1 + Q_2, \; P_2 = 30 + 2Q_1 - Q_2,$$

$$TC = 10Q_1 + Q_1Q_2 + 10Q_2, \; Q_1 + Q_2 = M = 15.$$

Then estimate the new optimal profit if $Q = M$ rises by 1 unit.

Maximize the profit function $\pi = P_1Q_1 + P_2Q_2 - TC$,

$$\pi = 40Q_1 - Q_1^2 + 2Q_1Q_2 + 20Q_2 - Q_2^2$$

subject to the constraint $Q_1 + Q_2 = M = 15$.
Result: $Q_1 = 10, Q_2 = 5$, $\lambda = 30$, and the maximum profit $\pi_{max} = 475$.
Since $\frac{\partial g}{\partial M} = \lambda$, so if M increases by 1 unit then g (also π) rises by λ units. The new optimal profit is $475 + 30 = 505$.
Remark: if the constraint increases by 1 unit, then the objective function rises by λ units.
* ppp 446

Maximize the Cobb-Douglas Production function using Lagrange multiplier method:

$$Q = Q(K, L) = AK^{\alpha}L^{\beta}$$

subject to a cost constraint

$$P_K K + P_L L = M.$$

Result: $K = \dfrac{\alpha M}{(\alpha+\beta)P_K}$, $L = \dfrac{\beta M}{(\alpha+\beta)P_L}$

* Using Lag. mul. method, we can check again a result in Sec. 5.5 that when Q max:

$$\frac{MP_L}{P_L} = \frac{MP_K}{P_K}.$$

With a cost constraint when firm maximizes output then the ratio of marginal product to price is the same for all inputs.

Key terms: page 449

*HW: Exercise 5.6, pages 450-, problems 2, 3, 4, 5, 6

HW: Exercise 5.6, pages 451-, problems 1, 2, 4, 6

THANK YOU VERY MUCH!