Chapter 7. Matrices

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Example: Suppose that a firm produces three types of good, G1, G2 and G3, which it sells to two customers, C1 and C2. The monthly sales for these goods are given in Table

		Monthly sales for goods		
		G1	G2	G3
Sold to	C1	7	3	4
customer	C2	1	5	6

The above information can be rewritten concisely in the form

$$A = \begin{bmatrix} 7 & 3 & 4 \\ 1 & 5 & 6 \end{bmatrix}$$

A (sales) matrix of order 2×3 (having 2 rows, 3 columns with 6 entries)

In general, a matrix of order $m \times n$ has m rows and n columns.
We denote matrices by capital letters: A, B, C...
The entry/ element on ith row and jth column of the matrix A is denoted by a_{ij}.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Some special matrixes:
- square matrix;

- the zero matrix θ;

- column vector, row vector;

Transposition

The transpose of an $m \times n$ matrix A is the $n \times m$ matrix denoted by A^T formed by turning rows into columns and vice versa.

Example:

$$\begin{bmatrix} 7 & 3 & 4 \\ 1 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 7 & 1 \\ 3 & 5 \\ 4 & 6 \end{bmatrix},$$

$$\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix}^T.$$

* ppp 488, 490

Example: Let the following matrices

$$A = \begin{bmatrix} 2 & 1 & 8 \\ -1 & 3 & -2 \end{bmatrix}, \; B = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 1 & 1 \end{bmatrix}.$$

Addition of matrix of equal size

What is $A + B$?

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$A = \begin{bmatrix} 2 & 1 & 8 \\ -1 & 3 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 1 & 1 \end{bmatrix}$.

Addition of matrix of equal size

What is $A + B$?

$$A + B = \begin{bmatrix} 5 & 1 & 10 \\ 1 & 4 & -1 \end{bmatrix}$$

Scalar multiplication

What are $2A$, $-3B$, $A - B$?

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Addition of matrix of equal size

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Scalar multiplication

What are $2A$, $-3B$, $A - B$?

$$2A = \begin{bmatrix} 4 & 2 & 16 \\ -2 & 6 & -4 \end{bmatrix}$$

*ppp 492, 494

Matrix multiplication

Example: Let

$$p = \begin{bmatrix} 50 & 30 & 20 \end{bmatrix}, \; q = \begin{bmatrix} 100 \\ 200 \\ 175 \end{bmatrix}.$$

Then we can multiply p and q by:

$$pq = 50 \times 100 + 30 \times 200 + 20 \times 175 = 14500$$

Matrix multiplication

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$$p = \begin{bmatrix} 50 & 30 & 20 \end{bmatrix}, \; q = \begin{bmatrix} 100 \\ 200 \\ 175 \end{bmatrix}.$$

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Example: What is

$$\begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & -2 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ -2 & 3 \\ 3 & -1 \end{bmatrix}$$

Matrix multiplication

Example: Let

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Then we can multiply p and q by:

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Example: What is

$$\begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & -2 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ -2 & 3 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -13 & 12 \end{bmatrix}.$$

If A is an $m \times s$ matrix and B is an $s \times n$ matrix then $C = AB$ is an $m \times n$ matrix and c_{ij} is found by multiplying the ith row of A into the jth column of B.

*ppp 497, 499, 500; *HW: Exercise 7.1, page 504-505

Identity matrix: I_n

Definition

Let A be an $n \times n$ square matrix. If there exists an $n \times n$ square matrix B such that $AB = BA = I_n$, then we say that A is invertible or nonsingular and B is called the inverse matrix of A, denoted by A^{-1}.

Example: Let $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, then we can check that

$$A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

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$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \times \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Inverse matrices of size 2×2

Example: find the inverse matrix of $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

Set $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We need $AB = BA = I_2$, therefore

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This leads to the following set of equations:

$$\begin{cases} 2a + 5c & = 1 \\ 1a + 3c & = 0 \\ 2b + 5d & = 0 \\ 1b + 3d & = 1 \end{cases}$$

and so we get $a = 3, c = -1, b = -5, d = 2$.

Hence $B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = A^{-1}$. (check again $AB = BA = I$)

Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

We define the determinant of A, denoted by $det(A)$ (or $|A|$) by

$$det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

Theorem

Suppose A is an $n \times n$ matrix. A is invertible (or non-singular) iff $det(A) \neq 0$.
In particular with $n = 2$,

$$A^{-1} = \frac{1}{det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

*ppp 510

Example: let the following matrices $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$,

$B = \begin{bmatrix} -1 & 3 & 5 \\ 2 & 4 & -2 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 2 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}$, and $b = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Find the matrices X, Y and Z satisfy $AX = b$, $AY = B$, $ZA = C$.

Example: let the following matrices $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$,

$B = \begin{bmatrix} -1 & 3 & 5 \\ 2 & 4 & -2 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 2 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}$, and $b = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Find the matrices X, Y and Z satisfy $AX = b$, $AY = B$, $ZA = C$.

Suggestion: Check that $det(A) \neq 0$, therefore A is invertible. Computing A^{-1} and then $X = A^{-1}b$, $Y = A^{-1}B$, $Z = CA^{-1}$.

Example: page 511; ppp 512

Cofactor

Let the matrix $A = [a_{ij}]$. The cofactor A_{ij} corresponding to the entry a_{ij} is defined to be the determinant of the matrix obtained by deleting row i and column j of A, prefixed by $(-1)^{i+j}$.

Example: find all the cofactors of the matrix

$$A = \begin{bmatrix} -1 & -2 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 4 \end{bmatrix}$$

*ppp 517

$$A = \begin{bmatrix} -1 & -2 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 4 \end{bmatrix}$$

The cofactors of the matrix are:

$$A_{11} = \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} = 2,\ \ A_{12} = -\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = -5,\ \ A_{13} = \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 1$$

$$A_{21} = -\begin{vmatrix} -2 & 2 \\ 2 & 4 \end{vmatrix} = 12, A_{22} = \begin{vmatrix} -1 & 2 \\ 3 & 4 \end{vmatrix} = -10, A_{23} = -\begin{vmatrix} -1 & -2 \\ 3 & 2 \end{vmatrix} =$$

$$A_{31} = \begin{vmatrix} -2 & 2 \\ 1 & 1 \end{vmatrix} = -4,\ \ A_{32} = -\begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} = 5,\ \ A_{33} = \begin{vmatrix} -1 & -2 \\ 2 & 1 \end{vmatrix} = 3$$

Remark: The matrix $[A_{ij}]$ is called the **adjugate matrix**.
$A^* = [A_{ij}]^T$ is called the **adjoint matrix**.

Determinant of a matrix

Let

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

We define

$$det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

We can expand the determinant along any row, or any column.

Example: find the determinant of the following matrices

$$A = \begin{bmatrix} -1 & -2 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 & -2 \\ 0 & 2 & 0 \\ 3 & 2 & 4 \end{bmatrix}$$

*Examples and ppp 518, 519

Inverse matrices of size 3×3

If $det(A) \neq 0$, then

$$A^{-1} = \frac{1}{det(A)} A^*,$$

where A^* is the adjoint matrix.

Example: find the inverse matrix (if it exists) of

$$A = \begin{bmatrix} -1 & -2 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 4 \end{bmatrix}$$

Solution:

$det(A) = -1 \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} - (-2) \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} + 2 \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} =$
$-1 \times 2 + 2 \times 5 + 2 \times 1 = 10 \neq 0$, so A is invertible.

The cofactors of the matrix are:

$$A_{11} = 2, \; A_{12} = -5, \; A_{13} = 1$$

$$A_{21} = 12, A_{22} = -10, A_{23} = -4$$

$$A_{31} = -4, \; A_{32} = 5, \; A_{33} = 3$$

The adjoint matrix of A is

$$A^* = \begin{bmatrix} 2 & 12 & -4 \\ -5 & -10 & 5 \\ 1 & -4 & 3 \end{bmatrix}$$

The inverse matrix of A is

$$A^{-1} = \frac{1}{det(A)} A^* = \frac{1}{10} \begin{bmatrix} 2 & 12 & -4 \\ -5 & -10 & 5 \\ 1 & -4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1/5 & 6/5 & -2/5 \\ -1/2 & -1 & 1/2 \\ 1/10 & -2/5 & 3/10 \end{bmatrix}$$

*ppp 521

Key Term: page 523

HW: Exercise 7.2, page 523-524

HW: Exercise 7.2*, page 525-526, problems 3-7

Matrix form of a linear system

Example: consider the system

$$\begin{cases} 2x_1 + x_2 - 3x_3 & = 1 \\ -x_1 + 3x_2 + 5x_3 & = 2 \end{cases}$$

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The system is equivalent to

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In general, we can express a linear system in the matrix form

$$AX = b,$$

where A is the matrix of coefficients, X is the column vector of unknown variables, b is the right-hand-side vector.

Example 1: (page 511) The equilibrium prices P_1 and P_2 for two goods satisfy the equations

$$\begin{cases} -4P_1 + P_2 & = -13 \\ 2P_1 - 5P_2 & = -7 \end{cases}$$

The matrix form is $AX = b$ with

$$A = \begin{bmatrix} -4 & 1 \\ 2 & -5 \end{bmatrix}, \; X = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}, \; b = \begin{bmatrix} -13 \\ -7 \end{bmatrix}$$

$$X = A^{-1}b = \tfrac{1}{18} \begin{bmatrix} -5 & -1 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} -13 \\ -7 \end{bmatrix} = \tfrac{1}{18} \begin{bmatrix} 72 \\ 54 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

*ppp 512

Example 2: (page 513) Consider the simple two-sector macroeconomic model:

$$\begin{cases} Y = & C + I^* \\ C = & aY + b \end{cases},$$

where $0 < a < 1$, $b > 0$. Express the system in matrix form to find the equilibrium levels of C and Y in term of a, b and I^*.

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Result:

$$Y = \frac{I^* + b}{1 - a}, \ C = \frac{aI^* + b}{1 - a}$$

*ppp 514

Example 3, (page 521): Determine the equilibrium prices of three interdependent commodities that satisfy

$$\begin{cases} 2P_1 + 4P_2 + P_3 & = 77 \\ 4P_1 + 3P_2 + 7P_3 & = 114 \\ 2P_1 + P_2 + 3P_3 & = 48 \end{cases}$$

*ppp 522

*HW: Exercise 7.2, pages 523-524
HW: Exercise 7.2, pages 525-526, problems 3, 6, 7, 9

Cramer's rule

The linear system $AX = b$ is called Cramer if A is a square matrix and $det(A) \neq 0$.
Then the system has a unique solution

$$x = A^{-1}b.$$

Moreover

$$x_j = \frac{det(A_j)}{det(A)}, j = 1, 2, \cdots, n,$$

where A_j is the matrix found from A by replacing the jth column by the right-hand-side vector.

Example 1: Solve the system

$$\begin{cases} 2x_1 + x_2 & = 4 \\ 3x_1 - 4x_2 & = -5 \end{cases}$$

Example 2: (page 528) Solve the system

$$\begin{bmatrix} -1 & 0 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -9 \\ 13 \end{bmatrix}$$

*ppp 530

Example 3: (page 530) Consider the three-sector macroeconomic mode

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -a & 0 \\ -1 & 0 & 1 & 1 \\ -t & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Y \\ C \\ Y_d \\ T \end{bmatrix} = \begin{bmatrix} I^* + G^* \\ b \\ 0 \\ T^* \end{bmatrix}$$

Using Cramer's rule, $Y = \dfrac{det(A_1)}{det(A)}$.

In expanding along the 1-st column, we find that

$$det(A_1) = I^* + G^* + b - aT^*, \; det(A) = 1 - a + at$$

Hence

$$Y = \frac{det(A_1)}{det(A)} = \frac{I^* + G^* + b - aT^*}{1 - a + at}$$

Example 4, (page 533): The model of two trading nations are given by

$$Y_1 = C_1 + I_1^* + X_1 - M_1 \qquad Y_2 = C_2 + I_2^* + X_2 - M_2$$

$$C_1 = 0.8Y_1 + 200 \qquad C_2 = 0.9Y_2 + 100$$

$$M_1 = 0.2Y_1 \qquad M_2 = 0.1Y_2$$

$$X_1 = M_2, \; X_2 = M_1.$$

With a simple transposition, we obtain

$$\begin{bmatrix} 0.4 & -0.1 \\ -0.2 & 0.2 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 200 + I_1^* \\ 100 + I_2^* \end{bmatrix}$$

$$\Rightarrow Y_1 = \frac{50 + 0.2I_1^* + 0.1I_2^*}{0.06}$$

Then the multiplier $\frac{\partial Y_1}{\partial I_2^*} = \frac{0.1}{0.06} = \frac{5}{3}$ means an increase in I_2^* by 1 unit leads to an increase in Y_1 by 5/3 units.

Key terms: page 535

HW: Exercise 7.3, pages 535-536

HW: Exercise 7.3*, pages 536-538, problems 1-5

THANK YOU VERY MUCH!