Chapter 9. Dynamics

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Content

Concepts of difference equations

Example: Consider national income in year t, which is denoted by Y_t. Suppose the growth rate of national income is 7% per year, then

$$Y_1 = 1.07Y_0,\ Y_2 = 1.07Y_1,\ Y_3 = 1.07Y_2, \ldots, Y_t = 1.07Y_{t-1}, \ldots$$

The relationship between Y_t and Y_{t-1} is given by the equation

$$Y_t = 1.07Y_{t-1},$$

called a difference equation.
It is well known that the sequence (Y_t) is a geo. prog. with the geo. ratio 1.07. Then the general term is

$$Y_t = Y_0\,1.07^t,$$

called the solution of the difference equation.

Definition: a difference equation (sometimes called a recurrence relation) is an equation that relates consecutive terms of sequence of numbers.

Example: a difference equation given by the sequence:

$$Y_0 = 3, \;\; Y_t = 2Y_{t-1}, t \geq 1.$$

The solution (the general term of the sequence) is

$$Y_t = Y_0 \, 2^t = 3 \times 2^t, t \geq 1.$$

The condition $Y_0 = 3$ is called the initial condition.
* ppp 578

The 1st order linear difference equation

Solve the difference equation

$$Y_t = bY_{t-1} + c, \; Y_0 = y_0 \tag{1}$$

Let U_t be the solution to the equation

$$U_t = bU_{t-1} \tag{2}$$

U_t is called the **complementary function** (denoted by CF).
Let y_t be an (any) **particular solution** (denoted by PS) of Eq. (1).
Then the **general solution** of Eq. (1) is of the form

$$Y_t = U_t + y_t = CF + PS.$$

Note that $CF = U_t = Ab^t$, for any constant A, and we choose

$$PS = y = \frac{c}{1-b} \text{ if } b \neq 1, \text{ or } PS = y = ct \text{ if } b = 1$$

*Don't forget to check the initial condition

Example 1: (page 579) Solve the difference equation

$$Y_t = 4Y_{t-1} + 21, \; Y_0 = 1$$

Step 1: find the complementary function $CF = A4^t$
Step 2: choose a particular solution (here $b = 4 \neq 1$):

$$PS = \frac{c}{1-b} = \frac{21}{1-4} = -7$$

Step 3: the general solution is

$$Y_t = CF + PS = A4^t - 7$$

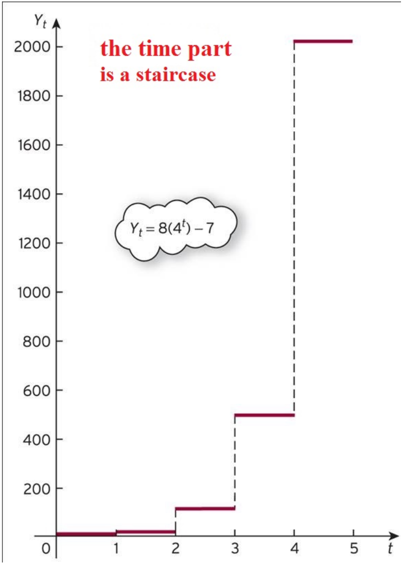
Step 4: check the initial condition

$$Y_0 = A4^0 - 7 = 1 \Rightarrow A = 8$$

Step 5: (conclude) the solution is

$$Y_t = 8 \times 4^t - 7$$

Comment on the qualitative behaviour of the solution
$Y_t = 8 \times 4^t - 7$



The Figure
shows that the values of
Y_t increase without bound as
t increases: since $b = 4 > 1$.

As $t \to +\infty$, $Y_t \to +\infty$

We say that
the time path of Y_t diverges uniformly or explodes.

Example 2: (page 579) Solve the difference equation

$$Y_t = \frac{1}{3} Y_{t-1} + 8, \; Y_0 = 2$$

Step 1: $CF = A(\frac{1}{3})^t$
Step 2: (here $b = \frac{1}{3} \neq 1$) $PS = \frac{c}{1-b} = \frac{8}{1-\frac{1}{3}} = 12$
Step 3: the general solution is

$$Y_t = CF + PS = A(\frac{1}{3})^t + 12$$

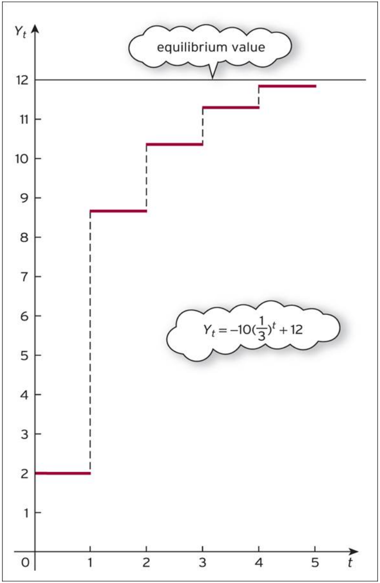
Step 4: check the initial condition

$$Y_0 = 2 \Leftrightarrow A + 12 = 2 \Leftrightarrow A = -10$$

Step 5: (conclude) the solution is

$$Y_t = -10(\frac{1}{3})^t + 12$$

Comment on the qualitative behavior of the solution
$Y_t = -10(\frac{1}{3})^t + 12$



The figure shows that the values of Y_t increase but eventually settle down at 12.

As $t \to +\infty$, $(\frac{1}{3})^t \to 0$, hence $Y_t \to 12$

We say that the time path of Y_t converges uniformly to 12, which is called the equilibrium value.

Example 3: Solve the difference equation

$$Y_t = Y_{t-1} + 7, Y_0 = 5$$

(here $b = 1,\ c = 7$)

The solution is

$$Y_t = Y_0 + ct = 5 + 7t$$

* ppp 582

Classifying solutions of difference equations

$$Y_t = bY_{t-1} + c,\ Y_0 = y_0 \qquad (1)$$

	General solution	Value of A to satisfy $Y_0 = y_0$
$b \neq 1$	$Y_t = Ab^t + c/(1-b)$	$A = y_0 - c/(1-b)$
$b = 1$	$Y_t = A + ct$	$A = y_0$

Analyzing the stability of model

	Y_t displays	The model is
$b > 1$	uniform divergence	unstable
$b < -1$	oscillatory divergence	unstable
$0 < b < 1$	uniform convergence	stable
$-1 < b < 0$	oscillatory convergence	stable

Difference equation for National income

As we have known the simple two-sector model: for $0 < a < 1$, $b > 0$,

$$Y = C + I, \quad C = aY + b, \quad I = I^*.$$

However in practice, there is a time lag between consumption and national income: consumption in period t, C_t, depends on national income in the previous period $(t-1)$, Y_{t-1}. Therefore we can rewrite the above relationships in the form

$$C_t = aY_{t-1} + b, \quad I_t = I^*, \quad Y_t = C_t + I_t.$$

Then we have

$$Y_t = aY_{t-1} + b + I^*.$$

This is a difference equation we have studied.

Example: (page 584) consider a two sector model

$$Y_t = C_t + I_t, \quad C_t = 0.8Y_{t-1} + 100, \quad I_t = 200.$$

Find an expression for Y_t when $Y_0 = 1700$. Is this system stable or unstable?

Example: (page 584) consider a two sector model

$$Y_t = C_t + I_t, \quad C_t = 0.8Y_{t-1} + 100, \quad I_t = 200.$$

Find an expression for Y_t when $Y_0 = 1700$. Is this system stable or unstable?
Substituting C_t and I_t into the 1st Eq. we get

$$Y_t = 0.8Y_{t-1} + 300, \;\; Y_0 = 1700.$$

After some steps, we find that the solution is

$$Y_t = 200 \times (0.8)^t + 1500.$$

Comment: because $0 < b = MPC = 0.8 < 1$, as t increases, $(0.8)^t$ converges to 0, hence Y_t converges to the equilibrium level 1500. The time path displays uniform convergence and the system is stable.
*ppp 584

Equilibrium price and commodity

However, for certain goods, there is a time lag between supply and price $\Rightarrow$ the supply Q_{S_t} in period t depends on the price P_{t-1} in the preceding period $(t-1)$.

Therefore we assume the time-dependent supply and demand equations as

$$\begin{aligned} Q_{S_t} &= aP_{t-1} - b \\ Q_{D_t} &= -cP_t + d \\ Q_{S_t} &= Q_{S_t} := Q_t \end{aligned}$$

Hence

$$P_t = -\frac{a}{c}P_{t-1} + \frac{b+d}{c}$$

Once a formula for P_t is obtained, we can deduce a corresponding formula for Q_t and analyze their time paths.

Example: (page 586) Consider the supply and demand equations

$$Q_{S_t} = 4P_{t-1} - 10, \quad Q_{D_t} = -5P_t + 35.$$

Assuming that the market is in equilibrium, find expressions for P_t and Q_t when $P_0 = 6$. Is the system stable or unstable?

Example: (page 586) Consider the supply and demand equations

$$Q_{S_t} = 4P_{t-1} - 10, \quad Q_{D_t} = -5P_t + 35.$$

Assuming that the market is in equilibrium, find expressions for P_t and Q_t when $P_0 = 6$. Is the system stable or unstable?
In equilibrium $Q_{D_t} = Q_{S_t} = Q_t$ then

$$5P_t + 35 = 4P_{t-1} - 10 \Leftrightarrow P_t = -0.8P_{t-1} + 9$$

After some steps, we find that

$$P_t = (-0.8)^t + 5 \Rightarrow Q_t = -5 \times (-0.8)^t + 10.$$

Comment: because $-1 < b = -0.8 < 0$, as t increases, $(-0.8)^t$ converges to 0, hence P_t and Q_t converge to the equilibrium levels 5 and 10, respectively. Both the time paths display oscillatory convergence and the system is stable.
*ppp 587

Key terms: page 588

*HW: Exercise 9.1, page 588-589, problems 1-5
HW: Exercise 9.1, page 588-589, problems 4,6

Concepts of differential equations

Example: It is known that

$$I(t) = \frac{dK}{dt}, \tag{3}$$

where $I(t)$ and K denote net investment flow and capital stock per time period. For any given function $I(t)$, $K = K(t)$ can be found by integrating $I(t)$ w.r.t t.
For example: if $I(t) = t$ then

$$K = \int I(t)dt = \int tdt = \frac{1}{2}t^2 + c, \quad \text{for any constant} \quad c \in \mathbb{R}.$$

The Eq. (3) is called a differential equation, and the above expression of K is called the general solution of Eq. (3).
If we assume an addition condition, for ex., $K(0) = k_0 = 500$ (initial condition), then c gets a specific value $c = 500$.

Definition: A differential equation (**DE**) is an equation that relates an (unknown) function with its derivatives.

A function usually represents a quantity, the derivatives represent the rates of change, and D.E. defines a relationship between them.

Example: consider the DE

$$\frac{du}{dt} = 3u.$$

The general solution is $u = Ae^{3t}$, for any value of constant A.

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$$\frac{du}{dt} = 3u.$$

The general solution is $u = Ae^{3t}$, for any value of constant A. If moreover the ini. cond. $u(0) = 5$ then $A = 5$, and so the solution is $u = 5e^{3t}$.

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Can you suggest the general solution to the equation

$$\frac{du}{dt} = mu, \quad \text{where } m \text{ is a constant?}$$

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Example: consider the DE

$$\frac{du}{dt} = 3u.$$

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Can you suggest the general solution to the equation

$$\frac{du}{dt} = mu, \quad \text{where } m \text{ is a constant?} \quad u = Ae^{mt}$$

* ppp 593

The 1st order linear DE

Solve the difference equation

$$\frac{dy}{dt} = my + c, \text{ with } y(0) = y_0. \tag{4}$$

Step 1: Solve the DE $\frac{du}{dt} = mu$:

$$u(t) = CF = Ae^{mt}$$

u is called the **complementary function** (denoted by CF).
Step 2: find an (any) **particular solution** (denoted by PS) of Eq. (4), namely $y^*(t)$. We choose:

1. if $m = 0$, $PS = y^*(t) = ct$;
2. if $m \neq 0$, $PS = y^*(t) = -\frac{c}{m}$. (we choose $y^*(t)$ as a constant function)

Step 3: the **general solution** of Eq. (4) is of the form

$$y(t) = CF + PS = u(t) + y^*(t)$$

Step 4: check the initial condition to specify a value for A
Step 5: conclude the solution $y(t)$

Example : (page 594) Solve the DE

$$\frac{dy}{dt} = -2y + 100,$$

with the initial condition:

$$(a)\ y(0) = 10;\quad (b)\ y(0) = 90;\quad (c)\ y(0) = 50$$

Example : (page 594) Solve the DE

$$\frac{dy}{dt} = -2y + 100,$$

with the initial condition:

$$(a)\ y(0) = 10; \quad (b)\ y(0) = 90; \quad (c)\ y(0) = 50$$

Here $m = -2$, and $c = 100$
Step 1: find the complementary function

$$CF = Ae^{-2t}$$

Step 2: choose a particular solution (here $b = -2 \neq 0$):

$$PS = -\frac{c}{m} = -\frac{100}{-2} = 50$$

Step 3: the general solution is

$$y(t) = CF + PS = Ae^{-2t} + 50$$

Step 4: check the initial condition

(a) $y(0) = 10$: $A + 50 = 10 \Rightarrow A = -40$;

(b) $y(0) = 90$: $A + 50 = 90 \Rightarrow A = 40$;

(c) $y(0) = 50$: $A + 50 = 50 \Rightarrow A = 0$.

Step 5: (conclude) the solution is

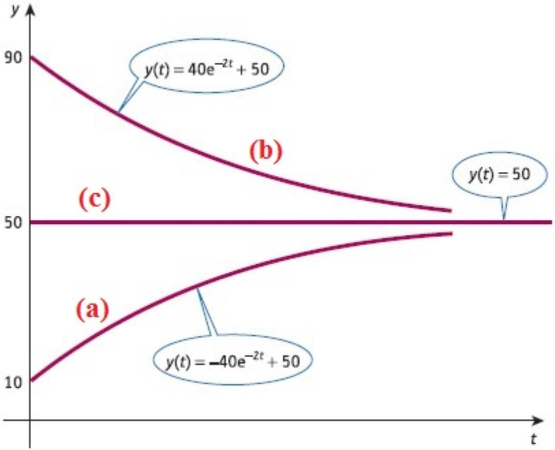
(a) $y(t) = -40e^{-2t} + 50$;

(b) $y(t) = 40e^{-2t} + 50$;

(c) $y(t) = 50$.

*ppp 597

Comment on the qualitative behavior of the solution



(a) The graph (the time part) of $y(t)$ increases from its initial value of 10 and settles down at the equilibrium value of 50 (also a PS) for sufficiently large t. Because $m = -2 < 0$, as t increases, e^{-2t} converges to 0, so $y(t)$ converges to 50.

(b) ... (by yourself)

(c) The graph of y is a horizontal line, the initial value of y equals to the he equilibrium value and y remains at this constant level for all time.

Classifying solutions and Stability of D.E.

$$\frac{dy}{dt} = my + c, \text{ with } y(0) = y_0. \qquad (4)$$

	General solution	Value of A to satisfy $Y_0 = y_0$
$m \neq 0$	$y(t) = Ae^{mt} - c/m$	$A = y_0 + c/m$
$m = 0$	$y(t) = A + ct$	$A = y_0$

Analyzing the stability of model

	y(t) displays	The model is
$m < 0$	convergence	stable
$m > 0$	divergence	unstable

DE for National income (page 598)

For the simple two-sector model, in practice, equilibrium is not immediately attained. Therefore we assume that the rate of change of Y w.r.t time is is proportional to the excess expenditure $C + I - Y$:

$$\frac{dY}{dt} = \alpha(C + I - Y),$$

for some positive adjustment coefficient α. We assume also

$$C = aY + b, \quad I = I^*, \; 0 < a < 1, b > 0.$$

Since these above relationships, we get a DE of standard form studied in this section:

$$\frac{dY}{dt} = \alpha(a-1)Y + \alpha(b + I^*)$$

Example: (page 599) consider the two sector model

$$\frac{dY}{dt} = 0.5(C + I - Y), \quad C = 0.8Y + 400, \quad I = 600.$$

Find an expression for $Y(t)$ when $Y(0) = 7000$. Is this system stable or unstable?

Example: (page 599) consider the two sector model

$$\frac{dY}{dt} = 0.5(C + I - Y), \quad C = 0.8Y + 400, \quad I = 600.$$

Find an expression for $Y(t)$ when $Y(0) = 7000$. Is this system stable or unstable?

Substituting C and I into the 1st Eq. we get

$$\frac{dY}{dt} = -0.1Y + 500$$

After some steps, we find that the solution is

$$Y(t) = 2000e^{-0.1t} + 5000$$

Comment: because $m = -0.1 < 0$, as t increases, $e^{-0.1t}$ converges to 0, hence $Y(t)$ converges to the equilibrium level 5000. The time path displays convergence and the system is stable.

*ppp 600

DE for Equi. price and commodity (page 600)

In the single-commodity market model,

$$Q_S = aP - b, \;\; Q_D = -cP + d, \;\; (a, b, c, d > 0),$$

we assume that the rate of change of price is proportional to excess demand $Q_D - Q_S$:

$$\frac{dP}{dt} = \alpha(Q_D - Q_S), \;\; \text{for some } \alpha > 0$$

Substituting Q_S and Q_D into 3rd-Eq. gives

$$\frac{dP}{dt} = -\alpha(a + c)P + \alpha(b + d),$$

which is a DE of standard form.

Example: (page 601) Consider the market model

$$Q_S = 3P - 4, \quad Q_D = -5P + 20, \quad \frac{dP}{dt} = \frac{1}{3}(Q_D - Q_S).$$

Find expressions for $P(t)$, $Q_S(t)$ and $Q_D(t)$ when $P(0) = 2$. Is the system stable or unstable?

Example: (page 601) Consider the market model

$$Q_S = 3P - 4, \;\; Q_D = -5P + 20, \;\; \frac{dP}{dt} = \frac{1}{3}(Q_D - Q_S).$$

Find expressions for $P(t)$, $Q_S(t)$ and $Q_D(t)$ when $P(0) = 2$. Is the system stable or unstable?

Substituting the expressions for Q_S and Q_D into 3rd-Eq. gives

$$\frac{dP}{dt} = -\frac{8}{3}6P + 8.$$

$CF = Ae^{-\frac{8}{3}t}$, $PS = -\frac{c}{m} = -\frac{8}{-\frac{8}{3}} = 3$, $P(t) = Ae^{-\frac{8}{3}t} + 3$.

$P(0) = 2 \Rightarrow A = -1$. The solution is $P(t) = -e^{-\frac{8}{3}t} + 3$.

$$Q_S(t) = 3P - 4 = -3e^{-\frac{8}{3}t} + 5,$$

$$Q_D(t) = -5P + 20 = 5e^{-\frac{8}{3}t} + 5.$$

Comment: because $m = -1.6 < 0$, as t increases, $e^{-1.6t}$ converges to 0, hence $P(t)$, $Q_S(t)$ and $Q_D(t)$ converge to the equilibrium levels 3, 5 and 5, respectively. The system is stable.

*ppp 602

Key terms: page 602

*HW: Exercise 9.2, page 603
HW: Exercise 9.2, page 604-605, problems 1, 3, 6

Formal mathematics: page 607

THANK YOU VERY MUCH!