CODE 1: FE-TH1-1920

FACULTY OF INFORMATION TECHNOLOGY
DEPARTMENT OF MATHEMATICS
School year 2019-2020

SOCIAL REPUBLIC OF VIETNAM
Independence - Freedom - Happiness
December 19, 2019

CALCULUS 1 (THE01001) - FINAL EXAM

Times: 75 minutes

Description of the test: this is a closed-book exam, any support material is not permitted; this test includes 5 problems with 8 questions. Points are distributed as follows:

Question	1	2	3	4	5	6	7	8
Points	1.0	1.0	1.0	1.5	1.5	1.5	1	1.5

Problem 1. Consider the linear map associated to the matrix $A = \begin{bmatrix} -3 & 2 \\ 2 & 1 \end{bmatrix}$.

a) $\boxed{1.0 \text{ pt}}$ Find the image of the vector $\begin{bmatrix} 5 \\ -2 \end{bmatrix}$ by the linear map.

b) $\boxed{1.0 \text{ pt}}$ Find the inverse matrix of A (if it exists).

Problem 2.

a) $\boxed{1.0 \text{ pt}}$ Suppose a vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ has length 9 and is 120^o counterclockwise form the positive x_2-axis. Find x_1 and x_2.

b) $\boxed{1.5 \text{ pt}}$ Use a rotation matrix to rotate the vector $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ counterclockwise by the angle 240^o.

Problem 3. $\boxed{1.5 \text{ pt}}$ Prove that the following limit does not exist

$$\lim_{(x,y)\to(0,0)} \frac{3xy}{2x^2+y^2}.$$

Problem 4. Let the function

$$f(x,y) = 4 - x^2 - y^2.$$

a) $\boxed{1.5 \text{ pt}}$ Determine the range and the level curves of f.

b) $\boxed{1.0 \text{ pt}}$ Find the tangent plane to the graph of f at the point $(1,-2,-1)$.

Problem 5. $\boxed{1.5 \text{ pt}}$ Find the Jacobi matrix for the vector-valued function

$$f(x,y) = \begin{bmatrix} x^2+2y^2 \\ e^{-x^2y^3} \end{bmatrix}.$$

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CODE 2: FE-TH1-1920

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Description of the test: this is a closed-book exam, any support material is not permitted; this test includes 5 problems with 8 questions. Points are distributed as follows:

Question	1	2	3	4	5	6	7	8
Points	1.0	1.0	1.0	1.5	1.5	1.5	1	1.5

Problem 1. Consider the linear map associated to the matrix $A = \begin{bmatrix} 2 & 4 \\ 3 & -3 \end{bmatrix}$.

a) $\boxed{1.0 \text{ pt}}$ Find the image of the vector $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$ by the linear map.

b) $\boxed{1.0 \text{ pt}}$ Find the inverse matrix of A (if it exists).

Problem 2.

a) $\boxed{1.0 \text{ pt}}$ Suppose a vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ has length 12 and is 30^o counterclockwise form the positive x_2-axis. Find x_1 and x_2.

b) $\boxed{1.5 \text{ pt}}$ Use a rotation matrix to rotate the vector $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ counterclockwise by the angle 210^o.

Problem 3. $\boxed{1.5 \text{ pt}}$ Prove that the following limit does not exist

$$\lim_{(x,y)\to(0,0)} \frac{5xy}{x^2 + 2y^2}.$$

Problem 4. Let the function

$$f(x, y) = 9 - x^2 - y^2.$$

a) $\boxed{1.5 \text{ pt}}$ Determine the range and the level curves of f.

b) $\boxed{1.0 \text{ pt}}$ Find the tangent plane to the graph of f at the point $(-2, 1, 4)$.

Problem 5. $\boxed{1.5 \text{ pt}}$ Find the Jacobi matrix for the vector-valued function

$$f(x, y) = \begin{bmatrix} 3x^2 + y^2 \\ e^{-x^3 y^2} \end{bmatrix}.$$

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SOLUTION FOR THE FINAL EXAM CODE 1: FE-TH1-1920

Problem 1.

a) $\boxed{1.0\text{ pt}}$ The image of the vector $\begin{bmatrix} 5 \\ -2 \end{bmatrix}$ by the linear map is

$$Ax = \begin{bmatrix} -3 & 2 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} -19 \\ 8 \end{bmatrix}.$$

b) $\boxed{1.0\text{ pt}}$ $det(A) = -7 \neq 0$ so A is invertible.

$$A^{-1} = \frac{1}{-7} \begin{bmatrix} 1 & -2 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} -1/7 & 2/7 \\ 2/7 & 3/7 \end{bmatrix}.$$

Problem 2.

a) $\boxed{1.0\text{ pt}}$ The vector is 120^o counterclockwise form the positive x_2-axis so it is $120^o + 90^o = 210^o$ counterclockwise form the positive x_1-axis, $\alpha = 210^o$. Then we have

$$x_1 = r\cos\alpha = 9\cos 210^o = -9\sqrt{3}/2,$$

$$x_2 = r\sin\alpha = 9\sin 210^o = -9/2.$$

b) $\boxed{1.5\text{ pt}}$ The angle $\alpha = 240^o$. The rotation matrix is

$$R_\alpha = \begin{bmatrix} \cos 240^o & -\sin 240^o \\ \sin 240^o & \cos 240^o \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{2} & \dfrac{\sqrt{3}}{2} \\ -\dfrac{\sqrt{3}}{2} & -\dfrac{1}{2} \end{bmatrix}.$$

Then the rotation of the vector is

$$\begin{bmatrix} -\dfrac{1}{2} & \dfrac{\sqrt{3}}{2} \\ -\dfrac{\sqrt{3}}{2} & -\dfrac{1}{2} \end{bmatrix} \times \begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{2+5\sqrt{3}}{2} \\ \frac{2\sqrt{3}-5}{2} \end{bmatrix}.$$

Problem 3. $\boxed{1.5\text{ pt}}$ Along the $x-$axis, $y = 0$: the limit becomes

$$\lim_{x\to 0} \frac{3x \times 0}{2x^2 + 0^2} = 0.$$

Along the straight line $x = y$: the limit becomes

$$\lim_{x=y\to 0} \frac{3x^2}{2x^2 + x^2} = \lim_{x=y\to 0} \frac{3x^2}{3x^2} = 1.$$

Along two different curves, the function approaches two different values $0 \neq 1$, therefore the limit does not exist.

Problem 4.

a) $\boxed{1.5 \text{ pt}}$ Since $x^2 \geq 0, y^2 \geq 0$ for any x, y so $f(x, y) \leq 4$. Hence the range of f is $R(f) = (-\infty, 4]$.

Now let c be a value in the range of f, $c \leq 4$, and we consider the equation

$$f(x, y) = 4 - x^2 - y^2 = c,$$

$$x^2 + y^2 = 4 - c.$$

If $c = 4$ then the above equation becomes

$$x^2 + y^2 = 0 \Leftrightarrow x = y = 0;$$

If $c < 4$ then the above equation defines a circle of center $(0, 0)$ and radius $\sqrt{4 - c}$.

Draw some particular level curves...

b) $\boxed{1.0 \text{ pt}}$ The partial derivatives are $f_x = -2x$, $f_y = -2y \Rightarrow f_x(1, -2) = -2$, $f_y(1, -2) = 4$. The equation of the tangent plane to the graph of f at the point $(1, -2, -1)$ is

$$z = -1 - 2(x - 1) + 4(y + 2),$$

$$z = -2x + 4y + 9.$$

Problem 5. $\boxed{1.5 \text{ pt}}$ The Jacobi matrix for the vector-valued function is

$$J(f) = \begin{bmatrix} (f_1)_x & (f_1)_y \\ (f_2)_x & (f_2)_y \end{bmatrix} = \begin{bmatrix} 2x & 4y \\ -2xy^3e^{-x^2y^3} & -3x^2y^2e^{-x^2y^3} \end{bmatrix}.$$

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SOLUTION FOR THE FINAL EXAM CODE 2: FE-TH1-1920

Problem 1.

a) $\boxed{1.0 \text{ pt}}$ The image of the vector $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$ by the linear map is

$$Ax = \begin{bmatrix} 2 & 4 \\ 3 & -3 \end{bmatrix} \times \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ -21 \end{bmatrix}.$$

b) $\boxed{1.0 \text{ pt}}$ $det(A) = -18 \neq 0$ so A is invertible.

$$A^{-1} = \frac{1}{-18} \begin{bmatrix} -3 & -4 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 1/6 & 2/9 \\ 1/6 & -1/9 \end{bmatrix}.$$

Problem 2.

a) $\boxed{1.0 \text{ pt}}$ The vector is 30^o counterclockwise form the positive x_2-axis so it is $30^o + 90^o = 120^o$ counterclockwise form the positive x_1-axis, $\alpha = 120^o$. Then we have

$$x_1 = r\cos\alpha = 12\cos 120^o = 12 \times (-1/2) = -6,$$

$$x_2 = r\sin\alpha = 12\sin 120^o = 12 \times \sqrt{3}/2 = 6\sqrt{3}.$$

b) $\boxed{1.5 \text{ pt}}$ The angle $\alpha = 210^o$. The rotation matrix is

$$R_\alpha = \begin{bmatrix} \cos 210^o & -\sin 210^o \\ \sin 210^o & \cos 210^o \end{bmatrix} = \begin{bmatrix} -\dfrac{\sqrt{3}}{2} & \dfrac{1}{2} \\ -\dfrac{1}{2} & -\dfrac{\sqrt{3}}{2} \end{bmatrix}.$$

Then the rotation of the vector is

$$\begin{bmatrix} -\dfrac{\sqrt{3}}{2} & \dfrac{1}{2} \\ -\dfrac{1}{2} & -\dfrac{\sqrt{3}}{2} \end{bmatrix} \times \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{3-4\sqrt{3}}{2} \\ \frac{-4-3\sqrt{3}}{2} \end{bmatrix}.$$

Problem 3. $\boxed{1.5 \text{ pt}}$ Along the $x-$axis, $y = 0$: the limit becomes

$$\lim_{x \to 0} \frac{5x \times 0}{x^2 + 2 \times 0^2} = 0.$$

Along the straight line $x = y$: the limit becomes

$$\lim_{x=y \to 0} \frac{5x^2}{x^2 + 2x^2} = \lim_{x=y \to 0} \frac{5x^2}{3x^2} = 5/3.$$

Along two different curves, the function approaches two different values $0 \neq 5/3$, therefore the limit does not exist.

Problem 4.

a) $\boxed{1.5 \text{ pt}}$ Since $x^2 \geq 0, y^2 \geq 0$ for any x, y so $f(x,y) \leq 9$. Hence the range of f is $R(f) = (-\infty, 9]$.

Now let c be a value in the range of f, $c \leq 9$, and we consider the equation

$$f(x,y) = 9 - x^2 - y^2 = c,$$

$$x^2 + y^2 = 9 - c.$$

If $c = 9$ then the above equation becomes

$$x^2 + y^2 = 0 \Leftrightarrow x = y = 0;$$

If $c < 9$ then the above equation defines a circle of center $(0,0)$ and radius $\sqrt{9-c}$.

Draw some particular level curves...

b) $\boxed{1.0 \text{ pt}}$ The partial derivatives are $f_x = -2x$, $f_y = -2y \Rightarrow f_x(-2,1) = 4$, $f_y(-2,1) = -2$. The equation of the tangent plane to the graph of f at the point $(-2,1,4)$ is

$$z = 4 + 4(x+2) - 2(y-1),$$

$$z = 4x - 2y + 14.$$

Problem 5. $\boxed{1.5 \text{ pt}}$ The Jacobi matrix for the vector-valued function is

$$J(f) = \begin{bmatrix} (f_1)_x & (f_1)_y \\ (f_2)_x & (f_2)_y \end{bmatrix} = \begin{bmatrix} 6x & 2y \\ -3x^2y^2e^{-x^3y^2} & -2x^3ye^{-x^3y^2} \end{bmatrix}.$$

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