CODE 1: FE-TH2-1819

FACULTY OF INFORMATION TECHNOLOGY
DEPARTMENT OF MATHEMATICS
School year 2018-2019

SOCIAL REPUBLIC OF VIETNAM
Independence - Freedom - Happiness
June 7, 2019

CALCULUS 2 (THE01002) - FINAL EXAM

Duration: 75 minutes

Description of the test : this is a closed-book exam, any support material is not permitted; this test includes 3 problems with 6 questions. Points are distributed as follows:

Question	1	2	3	4	5	6
Points	1.5	1.0	2.0	1.0	2.0	2.5

Problem 1. $\boxed{\text{1.5 pts}}$ Solve the differential equation

$$\frac{dy}{dx} = x^2y^2, \text{ with } y_0 = 1 \text{ if } x_0 = 1.$$

Problem 2. $\boxed{\text{6.0 pts}}$ Assume that the size of a population at time t, denoted by $N(t)$, evolves according to the logistic equation

$$\frac{dN}{dt} = 0.4N(1 - \frac{N}{200}), \text{ with } N(0) = 50.$$

a) $\boxed{\text{1.0 pt}}$ Find the intrinsic rate of growth and the carrying capacity.

b) $\boxed{\text{2.0 pts}}$ Find the equilibrium points of the equation and then analyze their stability, using both: the graphical approach, and the analytical approach.

c) $\boxed{\text{1.0 pt}}$ Find the population size at which the population grows fastest.

d) $\boxed{\text{2.0 pts}}$ Solve the differential equation with the given initial condition.

Problem 3. $\boxed{\text{2.5 pts}}$ Solve the system of linear differential equations

$$\begin{cases} x_1' = & 2x_1 + 3x_2 \\ x_2' = & 4x_1 + 3x_2 \end{cases}.$$

EDITED BY QUANG SANG PHAN

REVIEWED BY VU THU GIANG

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Problem 1. 1.5 pts Solve the differential equation

$$\frac{dy}{dx} = x^3 y, \text{ with } y_0 = 5 \text{ if } x_0 = 0.$$

Problem 2. 6.0 pts Assume that the size of a population at time t, denoted by $N(t)$, evolves according to the logistic equation

$$\frac{dN}{dt} = 0.5N(1 - \frac{N}{300}), \text{ with } N(0) = 50.$$

a) 1.0 pt Find the intrinsic rate of growth and the carrying capacity.

b) 2.0 pts Find the equilibrium points of the equation and then analyze their stability, using both: the graphical approach, and the analytical approach.

c) 1.0 pt Find the population size at which the population grows fastest.

d) 2.0 pts Solve the differential equation with the given initial condition.

Problem 3. 2.5 pts Solve the system of linear differential equations

$$\begin{cases} x_1' = & 2x_1 + 7x_2 \\ x_2' = & -5x_1 - 10x_2 \end{cases}.$$

Edited by Quang Sang Phan

Reviewed by Vu Thu Giang

SOLUTION FOR CODE 1: FE-TH2-1819

Problem 1. $\boxed{\text{1.5 pts}}$ $y \neq 0$ by the initial condition, then separate the variables we get

$$\frac{dy}{y^2} = x^2 dx$$

$$\int \frac{dy}{y^2} = \int x^2 dx$$

$$\frac{-1}{y} = \frac{x^3}{3} + C, \ C \in \mathbb{R}.$$

We check the initial condition $y(1) = 1$:

$$\frac{-1}{1} = \frac{1^3}{3} + C \Rightarrow \ C = \frac{-4}{3}.$$

Hence

$$\frac{-1}{y} = \frac{x^3}{3} - \frac{4}{3}.$$

$$y = \frac{3}{4 - x^3}.$$

Problem 2. $\boxed{\text{6.0 pts}}$

a) $\boxed{\text{1.0 pt}}$ From the equation, the intrinsic rate of growth $r = 0.4$ and the carrying capacity $K = 200$.

b) $\boxed{\text{2.0 pts}}$ We denote (the right hand side of the equation)

$$g(N) = 0.4N(1 - \frac{N}{200}).$$

Then

$$g(N) = 0 \Leftrightarrow \ N = 0 \text{ or } N = 200.$$

Find the equilibrium points are

$$N = 0 \text{ or } N = 200. \ \text{(0.5 pt)}$$

Analyze the stability:

- By the graphical approach 0.75 pt: First draw the graph of $g(N)$...
  (1) For $N = 0$: near $N = 0$, form the graph we see that if $N > 0$, the graph of $g(N)$ is above the axis $N = 0 \Rightarrow g(N) > 0$, i.e. $\frac{dN}{dt} = g(N) > 0$ then N is increasing. On the other hand when $N < 0$, the graph of $g(N)$ is under the axis $N = 0 \Rightarrow \frac{dN}{dt} = g(N) < 0$ then N is decreasing. The the equilibrium point $N = 0$ is locally unstable.
  (2) By the same way we can prove that the equilibrium point $N = 200$ is locally stable.
- analytical approach 0.75 pt: first compute

$$g'(N) = 0.4 - \frac{0.4N}{100}.$$

$g'(0) = 0.4 > 0$ so $N = 0$ is locally unstable; $g'(200) = -0.4 > 0$ so $N = 200$ is locally stable.

c) $\boxed{1.0\text{ pt}}$ The population grows fastest when the growth rate $\frac{dN}{dt} = g(N)$ is maximum. This happens when $N = 100$ (from the graph of $g(N)$ or from the equation $g'(N) = 0$).

d) $\boxed{2.0\text{ pts}}$ Solve the differential equation by separation of variables (see a general method in the pages 400-401 in the lecture book)

We rewrite the equation in the form

$$\frac{200dN}{N(N-200)} = -0.4dt.$$

Then

$$\int \frac{200dN}{N(N-200)} = \int -0.4dt.$$

$$\int (\frac{1}{N-200} - \frac{1}{N})dN = \int -0.4dt.$$

$$\ln \mid \frac{N-200}{N} \mid = -0.4t + C, \ C \in \mathbb{R}.$$

$$\mid \frac{N-200}{N} \mid = e^{-0.4t+C}$$

$$\frac{N-200}{N} = \pm e^{C} e^{-0.4t} = Me^{-0.4t}, \ M \in \mathbb{R}.$$

The initial condition $N(0) = 50$ leads to $M = -3$. Then

$$\frac{N-200}{N} = -3e^{-0.4t} \Rightarrow N = \frac{200}{1+3e^{-0.4t}}$$

Problem 3. $\boxed{2.5\text{ pts}}$ The corresponding matrix for the system is $A = \begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix}$.

The characteristic equation of A is

$$(2-\lambda)(3-\lambda) - 12 = 0 \Leftrightarrow \lambda^2 - 5\lambda - 6 = 0$$

$$\lambda = 6 \text{ or } \lambda = -1.$$

For $\lambda = 6$, we consider the homogeneous linear system

$$\begin{cases} -4u_1 + 3u_2 = & 0 \\ 4u_1 - 3u_2 = & 0 \end{cases} \Leftrightarrow u_1 = \frac{3}{4}u_2.$$

We choose an eigenvector $u = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

The same way for $\lambda = -1$, we choose an eigenvector $v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

The general solution for the system is

$$x = C_1 e^{6t} \begin{bmatrix} 3 \\ 4 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

$$\begin{cases} x_1 = & 3C_1e^{6t} - C_2e^{-t} \\ x_2 = & 4C_1e^{6t} + C_2e^{-t} \end{cases}.$$

SOLUTION FOR CODE 2: FE-TH2-1819

Problem 1. $\boxed{\text{1.5 pts}}$ $y \neq 0$ by the initial condition, then separate the variables we get

$$\frac{dy}{y} = x^3 dx$$

$$\int \frac{dy}{y} = \int x^3 dx$$

$$\ln|y| = \frac{x^4}{4} + C, \ C \in \mathbb{R}$$

$$|y| = e^{\frac{x^4}{4}+C}$$

$$y = \pm e^C e^{\frac{x^4}{4}} = Ke^{\frac{x^4}{4}}$$

We check the initial condition $y(0) = 5$:

$$K = 5.$$

Hence the solution is

$$y = 5e^{\frac{x^4}{4}}.$$

Problem 2. $\boxed{\text{6.0 pts}}$

a) $\boxed{\text{1.0 pt}}$ From the equation, the intrinsic rate of growth $r = 0.5$ and the carrying capacity $K = 300$.

b) $\boxed{\text{2.0 pts}}$ We denote (the right hand side of the equation)

$$g(N) = 0.5N(1 - \frac{N}{300}).$$

Then

$$g(N) = 0 \Leftrightarrow \ N = 0 \text{ or } N = 300.$$

Find the equilibrium points are

$$N = 0 \text{ or } N = 300. \ \ (0.5 \text{ pt})$$

Analyze the stability:

- By the graphical approach 0.75 pt: First draw the graph of $g(N)$...
  1. (1) For $N = 0$: near $N = 0$, form the graph we see that if $N > 0$, the graph of $g(N)$ is above the axis $N = 0 \Rightarrow g(N) > 0$, i.e. $\frac{dN}{dt} = g(N) > 0$ then N is increasing. On the other hand when $N < 0$, the graph of $g(N)$ is under the axis $N = 0 \Rightarrow \frac{dN}{dt} = g(N) < 0$ then N is decreasing. The the equilibrium point $N = 0$ is locally unstable.
  2. (2) By the same way we can prove that the equilibrium point $N = 300$ is locally stable.
- analytical approach 0.75 pt: first compute

$$g'(N) = 0.5 - \frac{0.5N}{150}.$$

$g'(0) = 0.5 > 0$ so $N = 0$ is locally unstable; $g'(300) = -0.5 > 0$ so $N = 300$ is locally stable.

c) $\boxed{1.0 \text{ pt}}$ The population grows fastest when the growth rate $\frac{dN}{dt} = g(N)$ is maximum. This happens when $N = 150$ (from the graph of $g(N)$ or from the equation $g'(N) = 0$).

d) $\boxed{2.0 \text{ pts}}$ Solve the differential equation by separation of variables (see a general method in the pages 400-401 in the lecture book)

We rewrite the equation in the form

$$\frac{300dN}{N(N-300)} = -0.5dt.$$

Then

$$\int \frac{300dN}{N(N-300)} = -0.5dt.$$

$$\int (\frac{1}{N-300} - \frac{1}{N})dN = \int -0.5dt.$$

$$\ln |\frac{N-300}{N}| = -0.5t + C, \; C \in \mathbb{R}$$

$$|\frac{N-300}{N}| = e^{-0.5t+C}$$

$$\frac{N-300}{N} = \pm e^{C} e^{-0.5t} = Me^{-0.5t}, \; M \in \mathbb{R}.$$

The initial condition $N(0) = 50$ leads to $M = -5$. Then

$$\frac{N-300}{N} = -5e^{-0.5t} \Rightarrow N = \frac{300}{1+5e^{-0.5t}}$$

Problem 3. $\boxed{2.5 \text{ pts}}$ The corresponding matrix for the system is $A = \begin{bmatrix} 2 & 7 \\ -5 & -10 \end{bmatrix}$.

The characteristic equation of A is

$$(2-\lambda)(-10-\lambda) + 35 = 0 \Leftrightarrow \lambda^2 + 8\lambda + 15 = 0$$

$$\lambda = -3 \text{ or } \lambda = -5.$$

For $\lambda = -3$, we consider the homogeneous linear system

$$\begin{cases} 5u_1 + 7u_2 = & 0 \\ -5u_1 - 7u_2 = & 0 \end{cases} \Leftrightarrow u_1 = -\frac{7}{5}u_2.$$

We choose an eigenvector $u = \begin{bmatrix} -7 \\ 5 \end{bmatrix}$.

The same way for $\lambda = -5$, we choose an eigenvector $v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

The general solution for the system is

$$x = C_1 e^{-3t} \begin{bmatrix} -7 \\ 5 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

$$\begin{cases} x_1 = & -7C_1 e^{-3t} - C_2 e^{-5t} \\ x_2 = & 5C_1 e^{-3t} + C_2 e^{-5t} \end{cases}.$$